

# Exact and Approximate Algorithms for Computing a Second Hamiltonian Cycle\*

Argyrios Deligkas<sup>†</sup>   George B. Mertzios<sup>‡</sup>   Paul G. Spirakis<sup>§</sup>   Viktor Zamaraev<sup>¶</sup>

## Abstract

In this paper we consider the following total functional problem: Given a cubic Hamiltonian graph  $G$  and a Hamiltonian cycle  $C_0$  of  $G$ , how can we compute a second Hamiltonian cycle  $C_1 \neq C_0$  of  $G$ ? Cedric Smith proved in 1946, using a non-constructive parity argument, that such a second Hamiltonian cycle always exists. Our main result is an algorithm which computes the second Hamiltonian cycle in time  $O(n \cdot 2^{(0.3-\varepsilon)n})$  time, for some positive constant  $\varepsilon > 0$ , and in polynomial space, thus improving the state of the art running time for solving this problem. Our algorithm is based on a fundamental structural property of Thomason’s lollipop algorithm, which we prove here for the first time. In the direction of approximating the length of a second cycle in a Hamiltonian graph  $G$  with a given Hamiltonian cycle  $C_0$  (where we may not have guarantees on the existence of a second Hamiltonian cycle), we provide a linear-time algorithm computing a second cycle with length at least  $n - 4\alpha(\sqrt{n} + 2\alpha) + 8$ , where  $\alpha = \frac{\Delta-2}{\delta-2}$  and  $\delta, \Delta$  are the minimum and the maximum degree of the graph, respectively. This approximation result also improves the state of the art.

**Keywords:** Hamiltonian cycle, cubic graph, exact algorithm, approximation algorithm.

## 1 Introduction

Graph Hamiltonicity problems are among the most fundamental problems in theoretical computer science. Problems related to Hamiltonian paths and Hamiltonian cycles have attracted a tremendous amount of work over the years, see for example the recent survey of Gould [10] and the references therein. Deciding whether a given graph has a Hamiltonian cycle, i.e. a cycle that contains each vertex once, was among Karp’s 21 NP-hard problems [12]. On the other hand, there are several exponential-time algorithms for computing a Hamiltonian cycle or a solution to the Traveling Salesman Problem (TSP), which is a direct generalization of the Hamiltonian cycle problem. The first algorithms for the problem were based on dynamic programming and required  $O(n^2 2^n)$  time [2, 11]. One of the next major improvements came decades later by Eppstein [7] who showed that a Hamiltonian cycle in a graph of degree at most three with  $n$  vertices can be computed in  $O(2^{\frac{n}{3}}) \approx 1.26^n$  time and linear space; at the same time the algorithm can also compute an optimum solution for TSP on such graphs. The algorithm of Eppstein works by forcing specific edges of the graph which must be part of any generated cycle; a variation of this algorithm can also enumerate all Hamiltonian cycles in a graph of degree at most three in  $O(2^{\frac{3n}{8}})$  time [7]. After that, there has been a series of improvements on the running time for TSP and the Hamiltonian cycle problem in degree-three graphs; the most recent results are an  $O(1.2553^n)$ -time algorithm by Liśkiewicz and Schuster [14] and an  $O^*(2^{\frac{3n}{10}}) \approx 1.2312^n$ -time algorithm by Xiao and Nagamochi [19], where  $O^*(\cdot)$  suppresses polynomial factors. By allowing exponential space, the

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<sup>†</sup>Department of Computer Science, Royal Holloway University of London, UK. Email: [argyrios.deligkas@rhul.ac.uk](mailto:argyrios.deligkas@rhul.ac.uk)

<sup>‡</sup>Department of Computer Science, Durham University, UK. Email: [george.mertzios@durham.ac.uk](mailto:george.mertzios@durham.ac.uk)

<sup>§</sup>Department of Computer Science, University of Liverpool, UK, and Computer Engineering & Informatics Department, University of Patras, Greece. Email: [p.spirakis@liverpool.ac.uk](mailto:p.spirakis@liverpool.ac.uk)

<sup>¶</sup>Department of Computer Science, University of Liverpool, Liverpool, UK. Email: [viktor.zamaraev@liverpool.ac.uk](mailto:viktor.zamaraev@liverpool.ac.uk)

running time for solving TSP on degree-three graphs can be improved further [4]. For general graphs there exists a randomized algorithm with running time  $O(1.657^n)$ , given by Bjorklund [3].

On the other hand, using a non-constructive parity argument, Cedric Smith and William Tutte [17] proved in 1946 that, for any fixed edge in a cubic (i.e. 3-regular) graph  $G$ , there exists an even (potentially zero) number of Hamiltonian cycles through this edge. Thus, the existence of a first Hamiltonian cycle guarantees the existence of a *second* one too, and this allows us to define the following total functional problem [15].

SMITH

**Input:** A cubic Hamiltonian graph  $G$  and a Hamiltonian cycle  $C_0$  of  $G$ .

**Task:** Compute a second Hamiltonian cycle  $C_1 \neq C_0$  of  $G$ .

It is easy to see that any algorithm  $\mathcal{A}$  for the Hamiltonian cycle (decision) problem on graphs with maximum degree three can be trivially adapted to solve SMITH as follows: for every edge  $e$  of the initial Hamiltonian cycle  $C_0$ , run  $\mathcal{A}$  on  $G \setminus e$ , i.e. on the graph obtained by removing  $e$  from  $G$ . Then, as a second Hamiltonian cycle  $C_1 \neq C_0$  always exists, at least one of these  $n$  calls of  $\mathcal{A}$  will return such a cycle  $C_1$ . That is, SMITH can be solved in  $n \cdot T(\mathcal{A})$  time, where  $T(\mathcal{A})$  is the worst-case running time of  $\mathcal{A}$  on input graphs with  $n$  vertices.

Thomason [16] was the first one who provided an algorithm, known as the *lollipop algorithm*, for SMITH. This algorithm starts from the given Hamiltonian cycle  $C_0$  of  $G$  and creates a sequence of distinct Hamiltonian paths where the last of these Hamiltonian paths trivially augments to a different Hamiltonian cycle of  $G$ . This algorithm was actually used by Papadimitriou to place SMITH within the complexity class PPA [15]. Although Thomason’s lollipop algorithm is well-known for decades, the internal structure of the algorithm’s execution on cubic Hamiltonian graphs remains so far mostly unclear and not well understood. In an attempt to construct worst-case instances for the lollipop algorithm, Cameron proved in 2001 [5] that on a specific family of cubic graphs (which is a variation of the family introduced by Krawczyk [13]) the lollipop algorithm runs in time at least  $2^{cn}$ , for some constant  $c$ . Thus, the state of the art running time for computing a second Hamiltonian cycle in SMITH is to use the best known algorithm for the Hamiltonian cycle problem in cubic graphs which runs in  $O^*(2^{\frac{3n}{10}})$  [19]. However, a tantalizing longstanding question is whether the knowledge of the first Hamiltonian cycle  $C_0$  *strictly helps* to reduce the running time for computing a second Hamiltonian cycle  $C_1$ . In this paper we provide evidence for the *affirmative* answer to this question.

A relaxation of SMITH is, given Hamiltonian cycle  $C_0$ , to efficiently compute a second cycle (different than  $C_0$ ) that is large enough. This relaxed problem becomes more meaningful for graphs with degrees larger than three, as it is well known that uniquely Hamiltonian graphs (i.e. graphs with a unique Hamiltonian cycle) exist, even when all vertices have degree three except two vertices which have degree four [6, 8]. For cubic Hamiltonian graphs, Bazgan, Santha, and Tuza [1] showed that the knowledge of the first Hamiltonian cycle  $C_0$  algorithmically strictly helps to approximate the length of a second cycle. In fact, if  $C_0$  is not given along with the input, there is no polynomial-time constant-factor approximation algorithm for finding a long cycle in cubic graphs, unless  $P=NP$ . In contrast, if  $C_0$  is given, then for every  $\varepsilon > 0$  a cycle  $C' \neq C_0$  of length at least  $(1-\varepsilon)n$  can be found in  $2^{O(1/\varepsilon^2)} \cdot n$  time, i.e. there is a linear-time PTAS for approximating the second Hamiltonian cycle [1]. The main ingredient in the proof of the latter result is an  $O(n^{\frac{3}{2}} \log n)$ -time algorithm which, given  $G$  and  $C_0$ , computes a cycle  $C' \neq C_0$  of length at least  $n - 4\sqrt{n}$  [1]. In wide contrast to cubic graphs, for graphs of minimum degree at least three, only existential proofs are known for a second large cycle. In particular, Girão, Kittipassorn, and Narayanan recently proved with a *non-constructive* argument that any  $n$ -vertex Hamiltonian graph with minimum degree at least 3 contains another cycle of length at least  $n - o(n)$  [9].

**Our contribution.** In this paper we do the first attempt to understand the internal structure of the lollipop algorithm of Thomason [16]. Our main result in this direction embarks from the following trivial observation, which is not specific to Thomason’s algorithm.

**Observation 1** *Let  $G$  be a cubic Hamiltonian graph and let  $C_0, C_1$  be any two different Hamiltonian cycles of  $G$ . Then the symmetric difference  $C_0 \Delta C_1$  of the edges of the two cycles is a 2-factor, i.e. a collection of cycles in  $G$ .*

Although Observation 1 determines that the symmetric difference of any two Hamiltonian cycles  $C_0$  and  $C_1$  is a collection of cycles in  $G$ , it does not rule out the possibility that  $C_0 \Delta C_1$  contains more than one cycle. Our first technical contribution is that, for any given Hamiltonian cycle  $C_0$ , there exists at least one other Hamiltonian cycle  $C_1$  such that  $C_0 \Delta C_1$  is *connected*, i.e. it contains exactly one cycle. More specifically, we prove that this holds for the particular Hamiltonian cycle  $C_1$  that is computed by Thomason's lollipop algorithm when starting from the cycle  $C_0$ . For our proof we simulate the execution of the lollipop algorithm by simultaneously assigning to every edge one of four distinct colors in a specific way such that four coloring invariants are maintained. Using this coloring procedure, an alternating red-blue path is maintained during the execution of the algorithm, which becomes an alternating red-blue *cycle* at the end of the execution. As it turns out, this alternating cycle coincides with the symmetric difference  $C_0 \Delta C_1$ .

This fundamental structural property of the lollipop algorithm (see Theorem 2 in Section 3) has never been revealed so far, and it enables us to design a novel and more efficient algorithm for detecting a second Hamiltonian cycle of  $G$ . This improves the current state of the art in the computational complexity of SMITH (see Section 4). Instead of trying to generate the second Hamiltonian cycle  $C_1$  directly from  $C_0$  (as Thomason's lollipop algorithm does), our new algorithm enumerates –almost– all alternating red-blue cycles, until it finds one alternating cycle  $D$  such that the symmetric difference  $C_0 \Delta D$  is a Hamiltonian cycle of  $G$  (and not just a collection of cycles that collectively contain all vertices of  $G$ ). During its execution, this algorithm iteratively has a choice between two different options for the next edge to be colored red, in which cases it branches to create two new instances. However, in order for the algorithm to achieve a strictly better worst-case running time than  $O^*(2^{\frac{3n}{10}})$ , it has to refrain from just always blindly branching to new instances. We are able to do this by identifying appropriate disjoint quadruples of edges, which we call *ambivalent quadruples*, and by *deferring* the choice for the colors of each of these quadruples until the very end. Then, at the last step of the algorithm we are able to choose their colors in nearly-linear time. That is, using the ambivalent quadruples we do not generate *all* possible alternating red-blue cycles but only a succinct representation of them. The running time of the algorithm that we achieve eventually is  $O(n \cdot 2^{(0.3-\varepsilon)n})$  time for some strictly positive  $\varepsilon > 0$ . In the particular case where the input graph  $G$  contains no induced cycle  $C_6$  on 6 vertices, the exponent becomes 0.2971925.

In the direction of approximating the length of a second cycle on graphs with minimum degree  $\delta$  and maximum degree  $\Delta$ , we provide in Section 5 a linear-time algorithm for computing a cycle  $C' \neq C_0$  of length at least  $n - 4a(\sqrt{n} + 2\alpha) + 8$ , where  $\alpha = \frac{\Delta-2}{\delta-2}$ . On the one hand, this improves the results of [1] in two ways. First, it provides a direct generalization to arbitrary Hamiltonian graphs of degree at least 3. Second, our algorithm works in linear time in  $n$  for all constant-degree regular graphs; in particular it works in time  $O(n)$  on cubic graphs (see Corollary 2). On the other hand, we complement the results of [9] as we provide a *constructive* proof for their result in case where the  $\Delta$  and  $\delta$  are  $o(\sqrt{n})$ -factor away from each other. Formally, our algorithm constructs in linear time another cycle of length  $n - o(n)$  whenever  $\frac{\Delta}{\delta} = o(\sqrt{n})$  (see Corollary 3).

## 2 Preliminaries

Given a graph  $G = (V, E)$ , an edge between two vertices  $u$  and  $v$  is denoted by  $uv \in E$ , and in this case  $u$  and  $v$  are said to be *adjacent* in  $G$ . The *neighborhood* of a vertex  $v \in V$  is the set  $N(v) = \{u \in V : uv \in E\}$  of its adjacent vertices. A graph  $G$  is cubic if  $|N(v)| = 3$  for every vertex  $v \in V$ . Given a path  $P = (v_1, v_2, \dots, v_k)$  (resp. a cycle  $C = (v_1, v_2, \dots, v_k, v_1)$ ) of  $G$ , the *length* of  $P$  (resp.  $C$ ) is the number of its edges. Furthermore,  $E(P)$  (resp.  $E(C)$ ) denotes the set of edges of the path  $P$  (resp. of the cycle  $C$ ). A path  $P$  (resp. cycle  $C$ ) in  $G$  is a *Hamiltonian* path (resp. *Hamiltonian* cycle) if it contains each vertex of  $G$  exactly once. Every cubic Hamiltonian graph is referred to as a *Smith graph*. Given a Smith graph  $G$  and a Hamiltonian cycle  $C_0$  of  $G$ , an edge of  $G$  which does not belong to  $C_0$  is called a *chord* of  $C_0$ , or simply a *chord*. The next theorem

allows us to assume without loss of generality that the input Smith graph  $G$  is triangle-free.

**Lemma 1** *Let  $G = (V, E)$  be a Smith graph with  $n$  vertices that contains at least one triangle, and let  $C_0$  be a Hamiltonian cycle of  $G$ . Then either there exists a second Hamiltonian cycle  $C_1$  of  $G$  that contains only two different edges than  $C_0$ , or there exists a Smith graph  $G'$  with  $n - 2$  vertices such that every Hamiltonian cycle in  $G$  corresponds bijectively to a Hamiltonian cycle in  $G'$  (or both).*

**Proof.** Let  $C_0 = (v_1, v_2, \dots, v_n)$ . As  $G$  contains at least one triangle, there must exist a vertex  $v_i$  such that  $v_{i-1}v_{i+1} \in E$ , i.e. the vertices  $v_{i-1}, v_i, v_{i+1}$  build a triangle. If  $v_iv_{i+2} \in E$ , then there exists a second Hamiltonian cycle  $C_1$  of  $G$ , in which the edges  $v_{i-1}v_i, v_iv_{i+1}, v_{i+1}v_{i+2}$  are replaced by the edges  $v_{i-1}v_{i+1}, v_{i+1}v_i, v_iv_{i+2}$ . Similarly, if  $v_iv_{i-2} \in E$ , then there exists a second Hamiltonian cycle  $C_1$  of  $G$ , in which the edges  $v_{i-2}v_{i-1}, v_{i-1}v_i, v_iv_{i+1}$  are replaced by the edges  $v_{i-2}v_i, v_iv_{i-1}, v_{i-1}v_{i+1}$ . In both cases, the second Hamiltonian cycle  $C_1$  of  $G$  contains only two different edges than  $C_0$ .

Now suppose that  $v_iv_{i+2}, v_iv_{i-2} \in E$ . We prove the statement of the lemma for the Smith graph  $G'$  that is obtained from  $G$  by removing vertices  $v_{i-1}$  and  $v_{i+1}$  and by connecting  $v_i$  to  $v_{i-2}$  and  $v_{i+2}$ . Let  $C$  be an arbitrary Hamiltonian cycle of  $G$ . If  $C$  contains the edges  $v_{i-1}v_{i-2}$  and  $v_{i-1}v_i$ , then it is easy to see that  $C$  must also contain the edges  $v_iv_{i+1}$  and  $v_{i+1}v_{i+2}$ , and thus  $C$  corresponds to a Hamiltonian cycle  $C'$  of  $G'$  in a straightforward way. Otherwise, if  $C$  contains the edges  $v_{i-1}v_{i-2}$  and  $v_{i-1}v_{i+1}$ , then it is easy to see that  $C$  must also contain the edges  $v_iv_{i+1}$  and  $v_iv_i^*$ , and thus in this case  $C$  also corresponds to a Hamiltonian cycle  $C'$  of  $G'$  in a straightforward way. The third case, where  $C$  contains the edges  $v_{i-1}v_i$  and  $v_{i-1}v_{i+1}$  is symmetric to the previous case. Thus every Hamiltonian cycle  $C$  of  $G$  corresponds to one Hamiltonian cycle  $C'$  of  $G'$ .

Conversely, let  $C'$  be a Hamiltonian cycle of  $G'$ . If  $C'$  contains the edges  $v_{i-2}v_i$  and  $v_iv_{i+2}$ , then we can create a Hamiltonian cycle  $C$  of  $G$  by replacing these edges with the edges  $v_{i-2}v_{i-1}, v_{i-1}v_i, v_iv_{i+1}, v_{i+1}v_{i+2}$ . If  $C'$  contains the edges  $v_{i-2}v_i$  and  $v_iv_i^*$ , then we can create a Hamiltonian cycle  $C$  of  $G$  by replacing these edges with the edges  $v_{i-2}v_{i-1}, v_{i-1}v_{i+1}, v_iv_{i+1}, v_iv_i^*$ . The third case where  $C'$  contains the edges  $v_iv_{i+2}$  and  $v_iv_i^*$  is symmetric to the previous case. Thus every Hamiltonian cycle  $C'$  of  $G'$  corresponds to one Hamiltonian cycle  $C$  of  $G$ . ■

The next theorem now follows immediately by repeatedly applying Lemma 1.

**Theorem 1** *Let  $G = (V, E)$  be a Smith graph with  $n$  vertices that contains at least one triangle, and let  $C_0$  be a Hamiltonian cycle of  $G$ . In linear time we can compute either a second Hamiltonian cycle  $C_1$  of  $G$  or a triangle-free Smith graph  $G'$  with fewer vertices such that every Hamiltonian cycle in  $G$  bijectively corresponds to a Hamiltonian cycle in  $G'$ .*

### 3 A connected symmetric difference of the two Hamiltonian cycles

In this section we present the fundamental structural property of Thomason's lollipop algorithm that the symmetric difference of the two involved Hamiltonian cycles is connected. For the sake of presentation, in this section we simulate Thomason's lollipop algorithm [16] on an arbitrary given Smith graph  $G$  and, during this simulation, we assign colors to some of the edges of  $G$ . In particular, we assign to some edges of  $G$  one of the colors *red*, *blue*, *black*, and *yellow*. Note that the colors of the edges change in every step of the lollipop algorithm. Furthermore, every such (partial) edge-coloring of  $G$  uniquely determines one step of the lollipop algorithm on  $G$  that starts at a specific initial configuration.

Thomason's lollipop algorithm starts (at Step 0) with a Hamiltonian cycle  $C_0 = (v_1, v_2, \dots, v_n, v_1)$ ; at this step we color all  $n$  edges of  $C_0$  *black*, while all other edges are colored *yellow*. Any Step  $i \geq 1$  of the lollipop algorithm is called *non-final* if the Hamiltonian path at this step does not correspond to a Hamiltonian cycle, i.e.  $v_1$  is not connected in  $G$  to the last vertex of this Hamiltonian path.

Step 1 is derived from Step 0 by *removing* the edge  $v_1v_n$  from the cycle  $C_0$ , thus obtaining the Hamiltonian path  $P_1 = (v_1, v_2, \dots, v_n)$ . We color this removed edge  $v_1v_n$  *red*. Let  $N(v_n) = \{v_1, v_{n-1}, v_k\}$ . At Step 2, the

lollipop algorithm continues by *adding* to the current Hamiltonian path  $P_1$  the edge  $v_n v_k$ , thus obtaining a “lollipop” in which  $v_k$  keeps all its three incident edges,  $v_1$  keeps only the incident edge  $v_1 v_2$ , and every other vertex keeps exactly two of its incident edges. Step 2 is completed by *removing* the edge  $v_k v_{k+1}$  from  $P_1$ , thus “breaking” the lollipop and obtaining the next Hamiltonian path  $P_2 = (v_1, v_2, \dots, v_k, v_n, \dots, v_{k+1})$ . It is important to note here that  $v_{k+1}$  is the vertex *immediately after* vertex  $v_k$  in the path  $P_{i-1}$ , where we consider that the path starts at  $v_1$ . At Step 2 we color the newly added edge  $v_n v_k$  *blue* and the removed edge  $v_k v_{k+1}$  *red*, while the last vertex of the path  $P_2$  is  $v_{k+1}$ . The algorithm continues towards Step 3 by adding to  $P_2$  the third edge incident to  $v_{k+1}$  (i.e. the unique incident edge  $v_{k+1} v_\ell$  different from the edges  $v_k v_{k+1}$  and  $v_{k+1} v_{k+2}$  that belonged to the previous path  $P_1$ ) and by removing again the other incident edge of  $v_\ell$  that “breaks” the lollipop. Similarly to Step 2, in Step 3 we color the newly added edge  $v_{k+1} v_\ell$  *blue* and the newly removed incident edge of  $v_\ell$  *red*.

As the lollipop algorithm progresses, the (partial) coloring of the edges of  $G$  continues, according to the following rules at Step  $i \geq 1$ . Recall that the Hamiltonian path at Step  $i \geq 1$  is denoted by  $P_i$ . Furthermore, assume that during Step  $i$ , the path  $P_i$  is obtained by *adding* to  $P_{i-1}$  the edge  $v_x v_y$  (where  $v_x$  is the last vertex of  $P_{i-1}$ , thus building a lollipop) and by subsequently *removing* from  $P_{i-1}$  the edge  $v_y v_z$ , thus breaking the constructed lollipop.

The description of the edge-coloring procedure that we apply at every step of the lollipop algorithm is formally given by the four coloring rules below. The intuitive description of these coloring rules is as follows. At every step, the black edges are those edges of the initial cycle  $C_0$  which are still contained in the current Hamiltonian path, while the red edges are all the remaining edges of  $C_0$ , i.e. those edges which do not belong to the current Hamiltonian path. The blue edges are those *chords* of  $C_0$  that belong to the current Hamiltonian path. Finally, the yellow edges are all the remaining chords of  $C_0$ , i.e. those chords that do not belong to the current Hamiltonian path. Initially we start with the cycle  $C_0$  that contains  $n$  black edges and we remove one of them (the edge  $v_1 v_n$ ) which becomes red. At every step of the algorithm we build the new *lollipop* when all three incident edges of some vertex  $v_y$  become either black or blue. This can happen either by adding a new (previously yellow) chord (thus coloring it blue) or by adding a new (previously colored red)  $C_0$ -edge (thus coloring it black). Once we have build the new lollipop, we break it within the same step of the lollipop algorithm, either by removing a (previously colored black)  $C_0$ -edge (thus coloring it red) or by removing a (previously colored blue) chord (thus coloring it yellow).

**Coloring Rule 1** *If  $v_x v_y \in C_0$  then we color  $v_x v_y$  black (in this case  $v_x v_y$  must be colored red at Step  $i-1$ ).*

**Coloring Rule 2** *If  $v_x v_y \notin C_0$  then we color  $v_x v_y$  blue (in this case  $v_x v_y$  must be yellow at Step  $i-1$ ).*

**Coloring Rule 3** *If  $v_y v_z \in C_0$  then we color  $v_y v_z$  red (in this case  $v_y v_z$  must be black at Step  $i-1$ ).*

**Coloring Rule 4** *If  $v_y v_z \notin C_0$  then we color  $v_y v_z$  yellow (in this case  $v_y v_z$  must be colored blue at Step  $i-1$ ).*

As we prove, the coloring of the edges proceeds such that the following invariants are maintained:

**Invariant 1** *During every non-final Step  $i \geq 1$ , every  $C_0$ -edge is colored either black or red.*

**Invariant 2** *During every non-final Step  $i \geq 1$ , every non- $C_0$ -edge is either blue or yellow.*

**Invariant 3** *At the end of every non-final Step  $i \geq 1$ , the set of all black and blue edges form a Hamiltonian path of  $G$ .*

**Invariant 4** *When the lollipop is built during any non-final Step  $i \geq 2$ , the set of all red and blue edges form an alternating path of even length in  $G$ , starting at  $v_1$  with a red edge. Furthermore, at the final step (i.e. when we build a second Hamiltonian cycle instead of a lollipop) the set of all red and blue edges form an alternating cycle  $D$  in  $G$ .*

The next observations follow immediately from the Coloring Rules 1-4 and by the proof of correctness of the Thomason’s lollipop algorithm [16].

**Observation 2** *Invariants 1 and 2 are maintained at every non-final Step  $i \geq 1$  of Thomason's lollipop algorithm.*

**Observation 3** *At every non-final Step  $i \geq 1$ , the set of black and blue edges are exactly the edges of the Hamiltonian path at this step of Thomason's lollipop algorithm, and thus Invariant 3 is maintained.*

In the next theorem we prove the maintenance of Invariant 4, which is our main technical contribution in this section.

**Theorem 2** *Invariant 4 is maintained at every (final or non-final) Step  $i \geq 1$  of Thomason's lollipop algorithm. Thus, after the final step of the algorithm, the symmetric difference  $C_0 \Delta C_1$  of  $C_0$  with the produced Hamiltonian cycle  $C_1$  is the alternating red-blue cycle  $D$ .*

**Proof.** The proof is done by induction on the number of steps of the lollipop algorithm. Invariant 4 is clearly true at Step 2 of the algorithm. In fact, when the lollipop is built during Step 2, there is only one red edge (i.e. the edge  $v_1v_n$ ) and one blue edge that is incident to  $v_n$ , thus creating together a red-blue alternating path of length 2. In the induction hypothesis, assume that Invariant 4 is true until the non-final Step  $i \geq 2$ . Let  $v_x$  be the last vertex of the Hamiltonian path  $P_{i-1}$  and let  $v_xv_y$  be the newly added edge that creates the lollipop during Step  $i$  (recall that  $v_xv_y$  is either blue or black). Furthermore, let  $v_yv_z$  be the edge of the path  $P_{i-1}$  that is removed in order to break the lollipop, i.e.  $v_yv_z$  is either a previously black  $C_0$ -edge that is colored red, or a previously blue chord that is yellow. Recall here that, by construction of the lollipop algorithm,  $v_z$  is the vertex *immediately after* vertex  $v_y$  in the path  $P_{i-1}$ , where the path starts at  $v_1$ . Then  $v_z$  becomes the last vertex of the Hamiltonian path  $P_{i+1}$  at Step  $i+1$ . Finally, let  $v_zv_w$  be the newly added (blue or black) edge that creates the lollipop during Step  $i+1$ . Note that  $v_w = v_1$  if and only if Step  $i+1$  is the final step.

*Case 1.*  $v_xv_y$  is a newly added *blue* edge at Step  $i$ , i.e. a blue chord. Then, by the induction hypothesis, none of the other two incident edges of  $v_y$  is blue at Step  $i$ , and thus they are both black as they both belong to the path  $P_{i-1}$ . Therefore  $v_y$  is the last vertex of the red-blue alternating path at the time that the lollipop is built at Step  $i$ . Furthermore, the edge  $v_yv_z$  is colored *red* when the lollipop is broken at Step  $i$ , according to Coloring Rule 3. Now we distinguish the two cases for the color of the newly added edge  $v_zv_w$  during Step  $i+1$ .

*Case 1a.*  $v_zv_w$  is colored *blue* during Step  $i+1$ , i.e.  $v_zv_w$  is a yellow chord at Step  $i$ . First let  $v_w \neq v_1$ , i.e. Step  $i+1$  is not a final step. Then, by the induction hypothesis, none of the other two incident edges of  $v_w$  is blue, and thus they are both black as they both belong to the path  $P_i$ . In this case, the alternating red-blue path at Step  $i$  (which ends at the vertex  $v_y$  with the blue edge  $v_xv_y$ ) is extended by the red edge  $v_yv_z$  and the blue edge  $v_zv_w$ . Now let  $v_w = v_1$ , i.e. Step  $i+1$  is the final step. Then, since until this step vertex  $v_1$  is an endpoint of the red-blue alternating path, the addition of the blue edge  $v_zv_w = v_zv_1$  at the final Step  $i+1$ , thus forming an alternating cycle (containing vertex  $v_1$  and the initial red edge  $v_1v_n$ ). This proves the induction step in Case 1a, see Figure 1(a).

*Case 1b.*  $v_zv_w$  is colored *black* during Step  $i+1$ , i.e.  $v_zv_w$  is a red  $C_0$ -edge at Step  $i$ . Then, since  $v_z$  is the vertex *immediately after* vertex  $v_y$  in the path  $P_{i-1}$ , it follows that  $v_z \neq v_1$ . First suppose that  $v_w = v_1$ , i.e. that Step  $i+1$  is the final step. Note that in this case  $v_z = v_n$ , since  $v_1v_n$  is the only red edge incident to  $v_1$  until Step  $i$ . Furthermore, since  $v_yv_z$  is colored *red* when the lollipop is broken at Step  $i$ , it follows that in this case  $v_y = v_{n-1}$ . Note that, by the induction hypothesis,  $v_z$  is incident to a blue chord at Step  $i$ , since  $v_z = v_n$  is the second vertex of the red-blue alternating path with even length. Thus, since  $v_yv_z$  becomes red at the end of Step  $i$  and  $v_zv_w$  becomes black at the final Step  $i+1$ , it follows that at Step  $i+1$  the red and blue edges form an alternating cycle (containing the red edge  $v_yv_z = v_{n-1}v_n$  and not containing vertex  $v_1$ ), thus proving the induction step, see the final step in Figure 1(b).

Now suppose (within Case 1b) that  $v_w \neq v_1$ , i.e. that Step  $i+1$  is non-final. Recall by the induction hypothesis that, until building the lollipop at Step  $i$ , the only red edge that is not incident to two blue edges is the first red edge, i.e.  $v_1v_n$ . Therefore, since  $v_z, v_w \neq v_1$  and  $v_zv_w$  is red at Step  $i$ , it follows that both  $v_z$

and  $v_w$  are incident to a blue chord at Step  $i$ . Thus, since at Step  $i$  the edge  $v_y v_z$  changes color from black to red, while at Step  $i + 1$  the edge  $v_z v_w$  changes its color from red to black, it follows by the induction hypothesis that during Step  $i + 1$  (when we break the lollipop) the red and blue edges form an alternating path in  $G$ , starting at  $v_1$  with a red edge and ending at  $v_w$  with a blue edge. This proves the induction step.

*Case 2.*  $v_x v_y$  is a newly added *black* edge at Step  $i$ , i.e. a  $C_0$ -edge that was previously colored red. Note that  $v_x, v_y \neq v_1$ , since Step  $i$  is not the final step by the induction hypothesis. Denote the three neighbors of  $v_y$  in  $G$  by  $v_x, v_z$ , and  $v_q$ , where  $v_z$  is the vertex *immediately after* vertex  $v_y$  in the path  $P_{i-1}$ . Since  $v_x$  is the last vertex of  $P_{i-1}$  and  $v_x v_y$  is red at the end of Step  $i - 1$ , it follows that one of the edges  $v_y v_z$  and  $v_y v_q$  is black and the other one is blue at the end of Step  $i - 1$ . Note that both edges  $v_y v_z$  and  $v_y v_q$  maintain their color after building the lollipop at Step  $i$ . Thus, since  $v_y$  has no red incident edge after building the lollipop at Step  $i$ , it follows by the induction hypothesis that vertex  $v_y$  is the last vertex of the red-blue alternating path.

*Case 2a.*  $v_y v_z$  is *blue* (and  $v_y v_q$  is black) at the beginning of Step  $i$ . Thus  $v_y v_z$  becomes yellow when we break the lollipop at Step  $i$ . Furthermore, since  $v_y v_z$  is a chord, the edge  $v_z v_w$  is a  $C_0$ -edge and it changes its color from red to black when we build the lollipop at Step  $i + 1$ . That is, the alternating red-blue path at Step  $i$  (which ends at  $v_z$  with the blue edge  $v_y v_z$ ) shrinks by removing from it the blue edge  $v_y v_z$  and the red edge  $v_z v_w$ .

Finally note that, if  $v_w = v_1$ , then we are left with no red and no blue edges after building the lollipop at Step  $i + 1$ , while Step  $i + 1$  is the final step. However, the absence of red and blue edges at the final step implies that the obtained Hamiltonian cycle of the lollipop algorithm is the same as the initial Hamiltonian cycle  $C_0$ , which is a contradiction to the correctness of the lollipop algorithm [16]. Therefore  $v_w \neq v_1$ , and thus Step  $i + 1$  is a non-final step. This completes the proof of the induction step in Case 2a.

*Case 2b.*  $v_y v_z$  is *black* (and  $v_y v_q$  is blue) after building the lollipop at Step  $i$ , i.e. the alternating red-blue path ends at  $v_z$  with the blue edge  $v_y v_q$ . Then the edge  $v_y v_z$  becomes *red* when we break the lollipop at the end of Step  $i$ . Furthermore recall that, at the time we build the lollipop at Step  $i + 1$ , the edge  $v_z v_w$  is either a chord that becomes blue (from yellow) or a  $C_0$ -edge that becomes black (from red).

First suppose that  $v_z v_w$  is a chord that becomes *blue* (from yellow) when we build the lollipop at Step  $i + 1$ . If  $v_w = v_1$  then Step  $i + 1$  is the final step, and in this case the alternating red-blue path becomes an alternating cycle (containing vertex  $v_1$  and the initial red edge  $v_1 v_n$ ) when we build the lollipop at Step  $i + 1$ , which proves the induction step. Now let  $v_w \neq v_1$ , i.e. Step  $i + 1$  is a non-final step. Then, since  $v_z v_w$  is a yellow chord at Step  $i$ , it follows by the induction hypothesis that  $v_w$  is not incident to any red edge when we build the lollipop at Step  $i + 1$ . Thus, the alternating red-blue path at Step  $i$  (which ends at  $v_y$  with the blue edge  $v_y v_q$ ) is augmented by adding to it the red edge  $v_y v_z$  and the blue edge  $v_z v_q$ , which again proves the induction step.

Now suppose that  $v_z v_w$  is a  $C_0$ -edge that becomes *black* (from red) when we build the lollipop at Step  $i + 1$ . Let  $v_w \neq v_1$  (i.e. Step  $i + 1$  is non-final). Thus, since at Step  $i$  the edge  $v_y v_z$  changes color from black to red, while at Step  $i + 1$  the edge  $v_z v_w$  changes its color from red to black, it follows by the induction hypothesis that during Step  $i + 1$  (when we break the lollipop) the red and blue edges form an alternating path in  $G$ , starting at  $v_1$  with a red edge and ending at  $v_w$  with a blue edge. This proves the induction step.

Finally let  $v_w = v_1$ , i.e. Step  $i + 1$  is the final step. Then  $v_z = v_n$ , since  $v_1 v_n$  is the only red edge incident to  $v_1$  until Step  $i$ . Furthermore, since  $v_y v_z$  is colored *red* when the lollipop is broken at Step  $i$ , it follows that in this case  $v_y = v_{n-1}$ . By the induction hypothesis,  $v_z$  is incident to a blue chord at Step  $i$ , since  $v_z = v_n$  is the second vertex of the red-blue alternating path with even length. Thus, since  $v_y v_z$  becomes red at the end of Step  $i$  and  $v_z v_w = v_n v_1$  becomes black at the final Step  $i + 1$ , it follows that at Step  $i + 1$  the red and blue edges form an alternating cycle (containing the red edge  $v_y v_z = v_{n-1} v_n$  and not containing vertex  $v_1$ ). This completes the proof of the induction step in Case 2b. ■

The next corollary follows by the proof of Theorem 2, and will allow us to reduce the asymptotic running time of our algorithm in Section 4 by a factor of  $n$ .

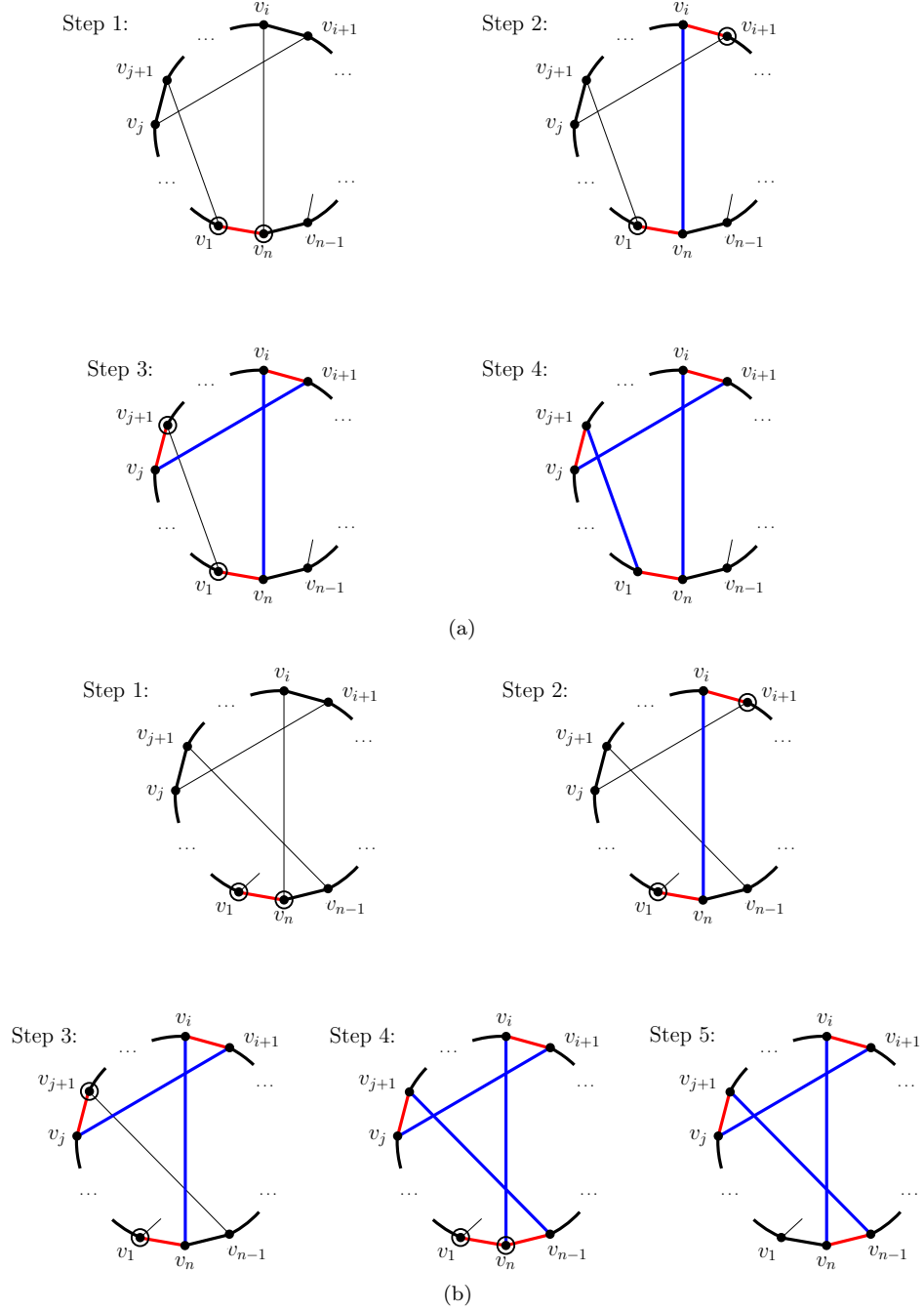


Figure 1: The edge-coloring during the execution of the lollipop algorithm in two example cubic Hamiltonian graphs. Every non-final Step  $i \geq 2$  of the algorithm encompasses both building the new lollipop (with the blue edge) and breaking it (with the red edge), thus the illustrated red and blue edges always have more one red edge (i.e. the last red edge) than the alternating red-blue path of even length (see Theorem 2). At every non-final Step  $i$  of the algorithm, the endpoints of the corresponding Hamiltonian path  $P_i$  are illustrated by a circled vertex. In the example (a), vertex  $v_1$  belongs to the alternating red-blue cycle, while in the example (b) vertex  $v_1$  does not belong to it (see the final step in each example).



**Corollary 1** *Let  $C_0$  be a given Hamiltonian cycle of a Smith graph  $G$ . Let  $(v_i, v_j, v_k)$  be three consecutive vertices of  $C_0$ . Then there exists a second Hamiltonian cycle  $C_1$  of  $G$  such that (i)  $C_0 \Delta C_1$  is a cycle in  $G$  and (ii) either the edge  $v_i v_j$  or the edge  $v_j v_k$  does not belong to  $C_1$ .*

**Proof.** Part (i) of the corollary follows immediately by the statements of Theorem 2 and of Invariant 4. To prove part (ii) of the corollary, first note that, due to symmetry, we may denote without loss of generality  $v_i = v_{n-1}$ ,  $v_j = v_n$ , and  $v_k = v_1$ .

Within the proof of Theorem 2, there are only two ways in which the alternating red-blue cycle can be built at the final step of Thomason’s lollipop algorithm. In the first way, the red-blue alternating cycle contains vertex  $v_1$  and the red edge  $v_1 v_n$  (this can happen only in Cases 1a and 2b). In the second way, the red-blue alternating cycle contains the red edge  $v_{n-1} v_n$  but not vertex  $v_1$  (this can happen only in Cases 1b and 2b). This completes the proof of the corollary. ■

## 4 The alternating cycles’ exploration algorithm

In this section we present our  $O(n \cdot 2^{(0.3-\varepsilon)n})$ -time algorithm for SMITH, where  $\varepsilon > 0$  is a strictly positive constant. This algorithm improves the state of the art, as it is asymptotically faster than all known algorithms for detecting a second Hamiltonian cycle in cubic graphs. Our algorithm is inspired from the structural property of Theorem 2. It starts from a designated vertex  $v_1$  and constructs an alternating cycle  $D$  of red-blue edges (with respect to  $C_0$ , in the terminology of Section 3) such that the symmetric difference  $C_0 \Delta D$  is a Hamiltonian cycle  $C_1$  of  $G$ . Equivalently, the algorithm constructs a second Hamiltonian cycle  $C_1$  such that the symmetric difference  $D = C_0 \Delta C_1$  is connected, i.e. one single cycle  $D$  of  $G$  in which every edge alternately belongs to  $C_0$  and to  $C_1$ , respectively.

Before we present and analyze our algorithm (Algorithm 1), we first present some necessary definitions and notation. Let  $G$  be a Smith graph and  $C_0 = (v_1, v_2, \dots, v_n)$  be the initial Hamiltonian cycle of  $G$ . For every vertex  $v_i$  of  $G$ , we denote by  $v_i^*$  the unique vertex that is connected to  $v_i$  through a chord. That is, whenever  $v_i v_j$  is a chord, we have that  $v_j = v_i^*$  and  $v_i = v_j^*$ . Furthermore, every vertex  $v_i$  is incident to exactly two  $C_0$ -edges  $v_{i-1} v_i$  and  $v_i v_{i+1}$ , where we consider all indices modulo  $n$ . Algorithm 1 iteratively *forces* specific edges to be colored *red* ( $C_0$ -edges not belonging to  $C_1$ ), *black* ( $C_0$ -edges belonging to  $C_1$ ), *blue* (chords belonging to  $C_1$ ), and *yellow* (chords not belonging to  $C_1$ ). Initially, the algorithm starts by coloring the  $C_0$ -edge  $v_1 v_t$  *red*, where  $v_t \in \{v_2, v_n\}$ , the chord  $v_t v_t^*$  *blue*, and the two  $C_0$ -edges adjacent to the edge  $v_t v_1$  *black*. That is, if  $v_t = v_2$  (resp. if  $v_t = v_n$ ) then the edges  $v_1 v_n$  and  $v_2 v_3$  (resp.  $v_1 v_2$  and  $v_{n-1} v_n$ ) are initially black. During its execution, the algorithm maintains an alternating red-blue path  $D$  of *even* length (starting with the red edge  $v_1 v_t$  and ending with a blue edge), until  $D$  eventually becomes an alternating cycle. Note that  $D$  can only become a cycle when we color the chord  $v_1 v_1^*$  blue. At every iteration the algorithm has (at most) two choices for the next red edge to be added to  $D$ , and thus it branches to (at most) two new instances of the problem, inheriting to both of them the choices of the forced (i.e. previously colored) edges made so far. At an arbitrary non-final step, let  $v_y$  be the last vertex of the alternating path  $D$ , and let  $v_x v_y$  be the last (blue) edge of  $D$ . For each of the two  $C_0$ -edges  $v_{y-1} v_y$  and  $v_y v_{y+1}$  that are incident to  $v_y$ , this edge is called *eligible* if it has not been forced (i.e. colored) at a previous iteration; otherwise it is called *non-eligible*. Here the term “eligible” stands for “eligible for branching”. We define the following operations; note that, once an edge has been assigned a color, it can *never* be forced to change its color.

- **Blue-Branch:** Whenever a chord  $v_x v_y$  is colored blue (where  $v_y$  is the last vertex of the current red-blue alternating path  $D$ ) and *both*  $C_0$ -edges  $v_y v_{y+1}, v_y v_{y-1}$  are eligible, we create two new instances  $I_1$  and  $I_2$ , where  $I_1$  (resp.  $I_2$ ) has the edge  $v_y v_{y+1}$  (resp.  $v_y v_{y-1}$ ) colored red and the edge  $v_y v_{y-1}$  (resp.  $v_y v_{y+1}$ ) colored black.
- **Blue-Force:** Whenever a chord  $v_x v_y$  is colored blue (where  $v_y$  is the last vertex of the current red-blue alternating path  $D$ ) and *exactly one* of the two  $C_0$ -edges  $v_y v_{y+1}, v_y v_{y-1}$  is eligible, we color this eligible  $C_0$ -edge red.

- **Red-Force:** Assume that a  $C_0$ -edge is colored red; note that this edge must be incident to a blue chord (i.e. its previous edge in the alternating path  $D$ ). If its other incident chord is uncolored, we color it blue. Otherwise, if it has been previously colored yellow, we announce “contradiction”. Moreover, if this new red edge is incident to a  $C_0$ -edge that is uncolored, we color this edge black.
- **Black-Force:** Assume that a  $C_0$ -edge  $v_i v_{i+1}$  is colored black, where this edge is adjacent to the (previously colored) black  $C_0$ -edge  $v_{i-1} v_i$  (resp.  $v_{i+1} v_{i+2}$ ). If their commonly incident chord  $v_i v_i^*$  (resp.  $v_{i+1} v_{i+1}^*$ ) is so far uncolored, we color it yellow. Otherwise, if it has been previously colored blue, we announce “contradiction”.
- **Yellow-Force:** Assume that a chord  $v_i v_i^*$  is colored yellow by the operation Black-Force (i.e. once both  $C_0$ -edges  $v_{i-1} v_i, v_i v_{i+1}$  become black); furthermore let  $v_k = v_i^*$ . If at least one of the  $C_0$ -edges  $v_{k-1} v_k, v_k v_{k+1}$  has been previously colored red, we announce “contradiction”. Otherwise, for each of the  $C_0$ -edges  $v_{k-1} v_k, v_k v_{k+1}$ , if this edge is uncolored, we color it black. (Note that, if the Yellow-Force operation does not announce “contradiction”, at the end of the operation all four  $C_0$ -edges  $v_{i-1} v_i, v_i v_{i+1}, v_{k-1} v_k, v_k v_{k+1}$  that are incident to the chord  $v_i v_i^*$  are colored black.)

The main idea of Algorithm 1 is as follows. In every non-final iteration we have that  $D = \text{Red} \cup \text{Blue}$  is an alternating *path*, while in the final iteration  $D$  is an alternating *cycle*. Suppose that, during a non-final iteration, we extend  $D$  by adding a new blue chord  $v_x v_y$  (where  $v_y$  is the last vertex of  $D$ ). The cases where not both edges  $v_y v_{y+1}, v_y v_{y-1}$  are eligible are covered by the following observation.

**Observation 4** *The only case, in which at least one of the  $C_0$ -edges  $v_y v_{y+1}, v_y v_{y-1}$  is red, is when  $v_y = v_1$ . In this case, exactly one of the  $C_0$ -edges  $v_y v_{y+1}, v_y v_{y-1}$  is red and the other one is black (see the initialization lines 1-5 of the algorithm), and thus  $D$  becomes an alternating cycle and the next iteration is the final one. In all other cases, where none of the  $C_0$ -edges  $v_y v_{y+1}, v_y v_{y-1}$  is red, at least one of them is eligible; otherwise both  $v_y v_{y+1}, v_y v_{y-1}$  are black, and thus their commonly incident chord  $v_x v_y$  has been previously colored yellow (by the operation Black-Force), a contradiction. If only one of these two edges is eligible, the algorithm is forced to color this edge red (with the operation Blue-Force).*

If both edges  $v_y v_{y+1}, v_y v_{y-1}$  are eligible, the algorithm *branches* (in most cases) to two new instances  $I_1$  and  $I_2$ , where  $I_1$  (resp.  $I_2$ ) has the eligible edge  $v_y v_{y+1}$  (resp.  $v_y v_{y-1}$ ) colored red. After the algorithm has branched to these two new instances  $I_1$  and  $I_2$ , it exhaustively applies the four forcing operations Blue-Force, Red-Force, Black-Force, and Yellow-Force, until none of them is applicable any more. The correctness of these forcing operations becomes straightforward by recalling our interpretation of the four colors, i.e. that the  $C_0$ -edges belonging (resp. not belonging) to  $C_1$  are colored *black* (resp. *red*), while the chords belonging (resp. not belonging) to  $C_1$  are colored *blue* (resp. *yellow*).

In some cases, the exhaustive application of the forcing rules in the two new instances  $I_1, I_2$  may only force very few edges, which results in a large running time of the algorithm before we reach a state where  $D$  becomes an alternating red-blue cycle. To circumvent this problem, we refrain from just always applying the operation Blue-Branch. Instead, in some cases we are able to *defer* the choice of the forced color of specific edges until the very end. More specifically, in some cases we are able to determine specific sets of 4 edges (each containing three  $C_0$ -edges and one chord) which build a  $C_4$  in  $G$  (i.e. a cycle of length 4) such that the all colored edges in the two different instances  $I_1, I_2$  are identical, apart from the colors of these 4 edges. Therefore all forcing operations in the subsequent iterations of the algorithm are *identical* in both these instances  $I_1, I_2$ , regardless of the specific colors of these 4 edges. Furthermore, as it turns out, every such a set of 4 of edges can receive forced colors in *exactly two* alternative ways. We call every such a set an *ambivalent quadruple* of edges. In these few cases, where an ambivalent quadruple occurs, we do not apply the operation Blue-Branch; instead we continue our forcing and branching operations in the subsequent iterations of the algorithm by only starting from one of these instances (instead of starting from both instances). Then, at the final step of the algorithm, i.e. when  $D$  becomes an alternating red-blue cycle, we are able to decide which of the two alternative edge colorings is correct for each ambivalent quadruple of edges (see the call to Procedure 2 in line 9 of the algorithm).

The above crucial trick of not always applying the operation Blue-Branch allows us to avoid generating *all* possible red-blue alternating cycles, thus obtaining an exponential speed-up of the algorithm and beating the state-of-the-art running time of  $O^*(2^{0.3n})$  which is implied by the TSP-algorithm of [20]. For example, in one of the cases where an ambivalent quadruple occurs, if we would branch to two new instances we would only force 5 new edges. Thus, since  $G$  has  $\frac{3}{2}n$  edges (as a cubic graph), forcing 5 edges at a time would imply the generation of at most  $O^*\left(2^{\frac{3}{2} \cdot \frac{1}{5}n}\right) = O^*(2^{0.3n})$  instances in the worst case, each of them corresponding to a different red-blue alternating cycle. However, by deferring the exact coloring of all ambivalent quadruples until the end of the algorithm we bypass this problem: instead of generating *all possible* red-blue alternating cycles, we create a succinct representation of them by only generating  $O(2^{(0.3-\varepsilon)n})$  alternating cycles (for some constant  $\varepsilon > 0$ ), and then we determine from them the desired alternating cycle, i.e. the one which gives us a second Hamiltonian cycle as its symmetric difference with the given first Hamiltonian cycle  $C_0$  (see the call to Procedure 2 in line 9 of the algorithm). Now we define operation Ambivalent-Flip, which appropriately changes at the end of the algorithm the already chosen colors of an ambivalent quadruple (see Procedure 2). Recall here that every ambivalent quadruple  $q$  contains exactly three  $C_0$ -edges and one chord.

- **Ambivalent-Flip:** Let  $q$  be an ambivalent quadruple of (already colored) edges. For every  $C_0$ -edge of  $q$ , if it has been colored red (resp. black), change its color to black (resp. red). Also, if the (unique) chord of  $q$  has been colored yellow (resp. blue), change its color to blue (resp. yellow).

Before we proceed with the proof of our main technical lemmas in this section (see Lemmas 2 and 3), we first need to define the notions of a *forcing path* and a *forcing cycle*. Intuitively, a forcing path consists of a sequence of edges of  $G$  such that, during the execution of Algorithm 1, once the first edge is forced to receive a specific color, every other edge of the path is also forced to receive some other specific color.

**Definition 1 (forcing path and cycle)** *Let  $G$  be a Smith graph. At an arbitrary iteration of Algorithm 1, a path  $P = (v_{i_1}, v_{i_2}, \dots, v_{i_k})$  of  $G$  is a forcing path starting at vertex  $v_{i_1}$  if:*

- *each of its edges  $v_{i_1}v_{i_2}, \dots, v_{i_{k-1}}v_{i_k}$  is yet uncolored and*
- *each of its first  $k-1$  vertices  $v_{i_1}, \dots, v_{i_{k-1}}$  is incident to exactly one already colored edge, while its last vertex  $v_{i_k}$  is incident to three yet uncolored edges.*

*Similarly, a cycle  $C = (v_{i_1}, v_{i_2}, \dots, v_{i_k}, v_{i_1})$  of  $G$  is a forcing cycle if:*

- *each of its edges  $v_{i_1}v_{i_2}, \dots, v_{i_{k-1}}v_{i_k}, v_{i_k}v_{i_1}$  is yet uncolored and*
- *each of its  $k$  vertices  $v_{i_1}, \dots, v_{i_k}$  is incident to exactly one already colored edge.*

Recall that, at every non-final iteration of the algorithm, there is exactly one *blue* edge  $v_xv_y$  such that its one endpoint  $v_x$  is incident to two other previously colored edges (one red and one black) and its other endpoint  $v_y$  is incident either to two uncolored edges or to one uncolored edge and one black edge. On the other hand, there might be several *black* edges  $v_iv_j$  such that  $v_i$  is incident to two other previously colored edges and  $v_j$  is incident to two uncolored edges. Furthermore, at the end of every iteration of the algorithm, every yellow and every red edge of  $G$  (apart from the first red edge of the alternating path  $D$ ) is adjacent to four other colored edges. Thus the next observation follows easily.

**Observation 5** *Let  $v_xv_y$  be the blue chord, where  $v_y$  is the last vertex of the red-blue alternating path  $D$  at some iteration of Algorithm 1. Furthermore let  $P = (v_y, \dots, v_\ell)$  be a forcing path of  $G$ , starting at vertex  $v_y$ . Then every internal vertex of  $P$  is incident to one black  $C_0$ -edge, as well as to one uncolored  $C_0$ -edge and to one uncolored chord.*

In the next lemma we prove the correctness of Algorithm 1. Next we prove our crucial technical Lemma 3 which specifies how the current instance is transformed in one iteration of our algorithm. The input instance  $I$  of the algorithm consists of a Smith graph  $G = (V, E)$ , a Hamiltonian cycle  $C_0$  of  $G$ , the set

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**Algorithm 1** ALTERNATING CYCLE DETECTION

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**Input:** Instance  $I = \{G, C_0, Q, Red, Blue, Black, Yellow\}$ , where  $G = (V, E)$  is a Smith graph,  $C_0 = (v_1, v_2, \dots, v_n)$  is an initial Hamiltonian cycle of  $G$ ,  $Q$  is a set of mutually disjoint quadruples of edges, and  $Red, Blue, Black, Yellow$  are four disjoint edge-subsets of  $E$  such that  $Red \cup Black \subseteq E(C_0)$  and  $Blue \cup Yellow \subseteq E \setminus E(C_0)$ .

**Output:** A second Hamiltonian cycle  $C_1$  of  $G$  such that  $D = C_0 \Delta C_1$  is connected.

```
1: if  $Q = Red = Blue = Black = Yellow = \emptyset$  then {initialization}
2:   Call the algorithm with the following parameters: {Look for an alternating cycle where  $v_1v_2$  is red}
3:    $Q \leftarrow \emptyset$ ;  $Red \leftarrow \{v_1v_2\}$ ;  $Blue \leftarrow \{v_2v_2^*\}$ ;  $Black \leftarrow \{v_2v_3, v_1v_n\}$ ;  $Yellow \leftarrow \emptyset$ 

4:   Call the algorithm with the following parameters: {Look for an alternating cycle where  $v_1v_n$  is red}
5:    $Q \leftarrow \emptyset$ ;  $Red \leftarrow \{v_1v_n\}$ ;  $Blue \leftarrow \{v_nv_n^*\}$ ;  $Black \leftarrow \{v_{n-1}v_n, v_1v_2\}$ ;  $Yellow \leftarrow \emptyset$ 
6: else {Main Iteration}
7:    $D \leftarrow Blue \cup Red$ 
8:   if  $D$  is a cycle then {final iteration of the algorithm}
9:     Call Procedure 2
10:  else { $D$  is a red-blue alternating path of even length}
11:    Let  $v_y$  be the last vertex of  $D$  (which ends with the blue chord  $v_xv_y$ )
12:    if  $v_yv_{y+1}, v_yv_{y-1} \notin Black$  then {both  $v_yv_{y+1}$  and  $v_yv_{y-1}$  are eligible  $C_0$ -edges}
13:      Let  $P^+$  and  $P^-$  be the forcing paths starting at  $v_y$  with the edge  $v_yv_{y+1}$  and  $v_yv_{y-1}$ , respectively
14:      if  $P^+ \cup P^-$  builds a  $C_4$  (i.e. a path with 4 edges) then
15:        Call Procedure 3
16:      else if  $P^+ \cup P^-$  builds a  $P_4$  (i.e. a path with 4 vertices) whose two endpoints are adjacent then
17:        Call Procedure 4
18:      else { $P^+ \cup P^-$  is neither a  $C_4$  nor a  $P_4$  whose two endpoints are adjacent}
19:        Apply the operation Blue-Branch, generating two new instances  $I_1, I_2$ 
20:        Apply exhaustively the operations Blue-Force, Red-Force, Black-Force, Yellow-Force to
        instances  $I_1, I_2$ , until no operation can be further applied
21:        for  $i \in \{1, 2\}$  do
22:          if no “contradiction” has been announced for  $I_i$  then call the algorithm on instance  $I_i$ 
23:        else {only one of  $v_yv_{y+1}, v_yv_{y-1}$  is an eligible  $C_0$ -edge}
24:          Apply exhaustively the operations Blue-Force, Red-Force, Black-Force, Yellow-Force to
          instance  $I$  until no operation can be further applied
25:          if no “contradiction” has been announced then
26:            Call the algorithm on the updated instance  $I$ 
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**Procedure 2** Check the alternating red-blue cycle  $D$ 

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1: for every ambivalent edge-quadruple  $q \in Q$  do
2:   if applying Ambivalent-Flip to the colors of  $q$  strictly reduces the number of connected components
   of  $C_0 \Delta D$  then
3:     Apply the operation Ambivalent-Flip to  $q$  and update  $D$  accordingly

4: if  $C_0 \Delta D$  is a Hamiltonian cycle of  $G$  then
5:   return Hamiltonian cycle  $C_1$  and alternating cycle  $D$ 
```

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$Q$  of all ambivalent quadruples, and four disjoint sets of forced (i.e. colored) edges  $Red, Blue, Black, Yellow$ . Initially the four sets of uncolored edges as well as the set  $Q$  are all empty. Given such an instance  $I = (G, C_0, Q, Red, Blue, Black, Yellow)$ , we denote by  $U(I) = E \setminus \{Red \cup Blue \cup Black \cup Yellow\}$  be the set of all *unforced* (i.e. uncolored) edges in this instance. Furthermore we denote by  $W(I)$  the set of vertices which are not incident to any edge of  $Red \cup Black$  in  $I$ ; we refer to the vertices of  $W(I)$  as *biased* vertices,

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**Procedure 3** Update of instance  $I$  when  $P^+ \cup P^-$  builds a  $C_4$

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- 1:  $Q \leftarrow Q \cup \{E(P^+ \cup E(P^-))\}$  {new ambivalent quadruple of edges}
  - 2:  $Red \leftarrow Red \cup \{v_y v_{y+1}\}; \quad Black \leftarrow Black \cup \{v_y v_{y-1}\}$
  - 3: Apply exhaustively the operations **Blue-Force**, **Red-Force**, **Black-Force**, **Yellow-Force** to instance  $I$  until no operation can be further applied
  - 4: **if** no “contradiction” has been announced **then**
  - 5:   Call Algorithm 1 on the updated instance  $I$
- 

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**Procedure 4** Create instances  $I_1, I_2$  when  $P^+ \cup P^-$  builds a  $P_4$  whose two endpoints are adjacent

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- 1:  $I_1 \leftarrow I; \quad I_2 \leftarrow I$
  - 2: **if**  $P^+$  contains only the edge  $v_y v_{y+1}$  and  $P^-$  contains the two edges  $v_y v_{y-1}, v_{y-1} v_{y+2}$  **then**
  - 3:   Update  $I_1$  such that:  
        $Red \leftarrow Red \cup \{v_y v_{y+1}\}; \quad Q \leftarrow Q \cup \{\{v_y v_{y+1}, v_y v_{y-1}, v_{y-1} v_{y+2}, v_{y+2} v_{y+1}\}\}$
  - 4:   Update  $I_2$  such that:  
        $Red \leftarrow Red \cup \{v_y v_{y-1}\}; \quad Black \leftarrow Black \cup \{v_{y+1} v_{y+2}\}$
  - 5: **else**  $\{P^+$  contains the two edges  $v_y v_{y+1}, v_{y+1} v_{y-2}$  and  $P^-$  contains only the edge  $v_y v_{y-1}\}$
  - 6:   Update  $I_2$  such that:  
        $Red \leftarrow Red \cup \{v_y v_{y-1}\}; \quad Q \leftarrow Q \cup \{\{v_y v_{y-1}, v_y v_{y+1}, v_{y+1} v_{y-2}, v_{y-2} v_{y-1}\}\}$
  - 7:   Update  $I_1$  such that:  
        $Red \leftarrow Red \cup \{v_y v_{y+1}\}; \quad Black \leftarrow Black \cup \{v_{y-1} v_{y-2}\}$
  - 8: Apply exhaustively the operations **Blue-Force**, **Red-Force**, **Black-Force**, **Yellow-Force** to instances  $I_1, I_2$ , until no operation can be further applied
  - 9: **for**  $i \in \{1, 2\}$  **do**
  - 10:   **if** no “contradiction” has been announced for  $I_i$  **then** call Algorithm 1 on instance  $I_i$
- 

while all other vertices in  $V - W(I)$  are referred to as *unbiased* vertices. Finally, we refer to the set of ambivalent quadruples of instance  $I$  as  $Q(I)$ .

**Lemma 2** *Let  $G = (V, E)$  be a Smith graph and  $C_0$  be a Hamiltonian cycle of  $G$ . Then, Algorithm 1 correctly computes a second Hamiltonian cycle  $C_1$  of  $G$  on the input  $I = (G, C_0, \emptyset, \emptyset, \emptyset, \emptyset)$ .*

**Proof.** First recall the interpretation of the four colors: the  $C_0$ -edges that belong (resp. do not belong) to the desired Hamiltonian cycle  $C_1$  are colored *black* (resp. *red*), while the chords that belong (resp. do not belong) to  $C_1$  are colored *blue* (resp. *yellow*). In the initialization phase, Algorithm 1 arbitrarily picks vertex  $v_1$  and it generates two instances  $I_1, I_2$  (see lines 1-5), where in  $I_1$  (resp.  $I_2$ ) the  $C_0$ -edge  $v_1 v_2$  is red, the chord  $v_2 v_2^*$  is blue, and the  $C_0$ -edges  $v_2 v_3, v_1 v_n$  are black (resp.  $v_1 v_n$  is red,  $v_n v_n^*$  is blue, and  $v_{n-1} v_n, v_1 v_2$  are black). Then the algorithm calls itself both on input  $I_1$  and on input  $I_2$ . This action of the algorithm is correct by Corollary 1, since the algorithm searches for the desired second Hamiltonian cycle  $C_1$  such that either  $v_1 v_2$  or  $v_1 v_n$  is red (i.e. does not belong to  $C_1$ ).

In each of these initial cases, the algorithm starts with one red  $C_0$ -edge and one blue chord (i.e. with an alternating red-blue path  $D$  of length two), and it iteratively extends  $D$  by adding one red  $C_0$ -edge and one blue chord to it, until either a “contradiction” is announced (by one of the forcing rules), or  $D$  becomes an alternating red-blue cycle by hitting the first red edge of  $D$  at vertex  $v_1$  with the blue chord  $v_1 v_1^*$ . In every non-final iteration of the algorithm, i.e. when  $D$  is still an alternating path, the algorithm proceeds as follows. Suppose that  $D$  ends at vertex  $v_y$  with the blue chord  $v_x v_y$ . Due to Observation 4, at most one of the two incident  $C_0$ -edges  $v_y v_{y+1}, v_y v_{y-1}$  is red. If exactly one of them is red, then the other incident  $C_0$ -edge is black, while  $D$  becomes an alternating cycle and the next iteration is the last one. Furthermore, it also follows by Observation 4 that, whenever none of the  $C_0$ -edges  $v_y v_{y+1}, v_y v_{y-1}$  is red, there is at least one eligible (i.e. uncolored) edge and at most one black edge among  $v_y v_{y+1}, v_y v_{y-1}$ .

If exactly one of these two edges is eligible and the other one is black (see lines 23-26), then the algorithm is forced to color this eligible edge red. Thus, in this case the algorithm correctly updates the current instance  $I$  by exhaustively applying the forcing operations Blue-Force, Red-Force, Black-Force, and Yellow-Force until none of them can be applied any more. Now suppose that both edges  $v_y v_{y+1}, v_y v_{y-1}$  are eligible, and let  $P^+$  (resp.  $P^-$ ) be the forcing path starting at vertex  $v_y$  with the edge  $v_y v_{y+1}$  (resp.  $v_y v_{y-1}$ ). Furthermore suppose that none of the conditions of lines 14 and 16 are satisfied. Then the algorithm first branches into two new instances  $I_1, I_2$  (by applying Blue-Force at vertex  $v_y$ ) and it then exhaustively applies the four forcing rules (see lines 18-22). The correctness of these forcing operations becomes straightforward by recalling our interpretation of the four colors. At the final iteration of the algorithm (see the call to Procedure 2 in line 9 of the algorithm), if the set  $Q$  of all ambivalent quadruples is empty, then the algorithm just checks whether the symmetric difference between  $C_0$  and the produced alternating red-blue cycle  $D$  is just one cycle. The correctness of this check follows by Corollary 1.

It remains to prove the correctness of the algorithm also in the case where one of the conditions of lines 14 and 16 is satisfied. First we analyze each of these two cases separately, as follows.

*Case 1: line 14 is applied.* The union  $P^+ \cup P^-$  of the two forcing paths builds a  $C_4$  (i.e. a cycle with 4 edges). Assume that each of the paths  $P^+, P^-$  has two edges, i.e.  $P^+$  contains the  $C_0$ -edge  $v_y v_{y+1}$  and the chord  $v_{y+1} v_{y+1}^*$ , while  $P^-$  contains the  $C_0$ -edge  $v_y v_{y-1}$  and the chord  $v_{y-1} v_{y-1}^*$ . Then, since  $P^+ \cup P^-$  is a  $C_4$ , it follows that  $v_{y+1}^* = v_{y-1}^*$ , and thus  $v_{y+1}^*$  is incident to two different chords, which is a contradiction as  $G$  is a cubic graph. Therefore, one of the forcing paths  $P^+, P^-$  has length 1 and the other one has length 3. As these two cases are symmetric, assume without loss of generality that  $P^+$  has length 1 (i.e. it only contains the  $C_0$ -edge  $v_y v_{y+1}$ ) and  $P^-$  has length 3. Since vertex  $v_{y+1}$  is the common endpoint of  $P^+$  and  $P^-$ , it follows that  $P^-$  contains the  $C_0$ -edge  $v_y v_{y-1}$ , the chord  $v_{y-1} v_{y+2}$ , and the  $C_0$ -edge  $v_{y+2} v_{y+1}$  (in this order). That is, the cycle  $P^+ \cup P^-$  contains exactly three  $C_0$ -edges and one chord. Furthermore note that the third edge incident to  $v_{y+1}$  (apart from  $v_y$  and  $v_{y+2}$ ) is the chord  $v_{y+1} v_{y+1}^*$ .

Applying the operation Blue-Branch at this iteration would result in the creation of two new instances  $I_1, I_2$ , in which the edges are colored as follows. In  $I_1$ , the  $C_0$ -edge  $v_y v_{y+1}$  is colored red, the  $C_0$ -edges  $v_y v_{y-1}, v_{y+2} v_{y+1}$  are colored black, and the chord  $v_{y-1} v_{y+2}$  is colored yellow. In  $I_2$ , the  $C_0$ -edge  $v_y v_{y+1}$  is colored black, the  $C_0$ -edges  $v_y v_{y-1}, v_{y+2} v_{y+1}$  are colored red, and the chord  $v_{y-1} v_{y+2}$  is colored blue. Note that, in *both* instances  $I_1, I_2$ , these edge colorings force the chord  $v_{y+1} v_{y+1}^*$  to be colored blue, and these are all edge colorings that can be forced so far.

That is, the only difference between the instances  $I_1, I_2$  is the way the four edges of  $P^+ \cup P^-$  are colored. Therefore, since each of the vertices of  $P^+ \cup P^-$  is “saturated” in both  $I_1, I_2$  (i.e. it has all its three incident edges colored), all forcing operations in the subsequent iterations of Algorithm 1 are *identical* in both  $I_1, I_2$ . Using this fact, the algorithm marks the edges of  $P^+ \cup P^-$  as an *ambivalent* quadruple of edges (see line 1 of Procedure 3). Furthermore, it colors these 4 edges according to  $I_1$  only, i.e. without branching to both  $I_1, I_2$ , and it calls itself on the updated instance  $I$  (see lines 2-5 of Procedure 3). Since all subsequent forcing operations would be identical in both instances  $I_1, I_2$ , the algorithm continues either until a contradiction is concluded at a later iteration (in this case a contradiction would be concluded by both  $I_1, I_2$ ) or until  $D$  becomes an alternating cycle.

*Case 2: line 16 is applied.* The union  $P^+ \cup P^-$  of the two forcing paths builds a  $P_4$  (i.e. a path with 4 vertices) whose two endpoints are adjacent. In this case, clearly one of the paths  $P^+, P^-$  contains one edge and the other one contains two edges. As these two cases are symmetric (see lines 2-4 and lines 5-7 of Procedure 4, respectively), it suffices to only analyze here the case that  $P^+$  contains only the  $C_0$ -edge  $v_y v_{y+1}$  and  $P^-$  contains the  $C_0$ -edge  $v_y v_{y-1}$  and the chord  $v_{y-1} v_{y+2}$ . Note that, by the assumption of Case 2, the endpoints of the paths  $P^+, P^-$  are connected via the  $C_0$ -edge  $v_{y+1} v_{y+2}$ . Furthermore, note that the third edge incident to  $v_{y+1}$  (apart from  $v_y$  and  $v_{y+2}$ ) is the chord  $v_{y+1} v_{y+1}^*$ ; similarly, the third edge incident to  $v_{y+2}$  (apart from  $v_{y-1}$  and  $v_{y+1}$ ) is the  $C_0$ -edge  $v_{y+2} v_{y+3}$ .

Applying the operation Blue-Branch at this iteration would result in the creation of two new instances  $I_1, I_2$ , in which the edges are colored as follows. In  $I_1$ , the  $C_0$ -edge  $v_y v_{y+1}$  is colored red, the  $C_0$ -edges  $v_y v_{y-1}, v_{y+1} v_{y+2}, v_{y+2} v_{y+3}$  are colored black, the chord  $v_{y-1} v_{y+2}$  is colored yellow, and the chord  $v_{y+1} v_{y+1}^*$  is colored blue. On the other hand, in  $I_2$  the  $C_0$ -edge  $v_y v_{y+1}$  is colored black, the  $C_0$ -edge  $v_y v_{y-1}$  is colored

red, and the chord  $v_{y-1}v_{y+2}$  is colored blue. Consider now applying again the operation Blue-Branch in the instance  $I_2$  at vertex  $v_{y+2}$  (once the chord  $v_{y-1}v_{y+2}$  has been colored blue in  $I_2$ ). This would replace instance  $I_2$  by two new instances  $I_2^1$  and  $I_2^2$ , in which the edges are colored as follows. In  $I_2^1$ , the  $C_0$ -edges  $v_yv_{y+1}, v_{y+2}v_{y+3}$  are colored black, the  $C_0$ -edges  $v_yv_{y-1}, v_{y+1}v_{y+2}$  are colored red, and the chords  $v_{y-1}v_{y+2}, v_{y+1}v_{y+1}^*$  are colored blue. Furthermore, in  $I_2^2$  the  $C_0$ -edges  $v_yv_{y+1}, v_{y+1}v_{y+2}$  are colored black, the  $C_0$ -edges  $v_yv_{y-1}, v_{y+2}v_{y+3}$  are colored red, the chord  $v_{y-1}v_{y+2}$  is colored blue, and the chord  $v_{y+1}v_{y+1}^*$  is colored yellow.

Now note that both instances  $I_1$  and  $I_2^1$  are identical, apart from the colors of the four edges  $v_yv_{y+1}, v_yv_{y-1}, v_{y-1}v_{y+2}, v_{y+1}v_{y+2}$ . Therefore, since each of these four vertices  $v_y, v_{y+1}, v_{y-1}, v_{y+1}$  is “saturated” in both  $I_1$  and  $I_2^1$  (i.e. it has all its three incident edges colored), all forcing operations in the subsequent iterations of Algorithm 1 are *identical* in both  $I_1, I_2^1$ . Thus, instead of branching into the three instances  $I_1, I_2^1, I_2^2$ , the algorithm only branches into the two instances  $I_1$  and  $I_2^2$  in Procedure 4, while it also marks the above four edges as an *ambivalent* quadruple of edges within  $I_1$  (see line 3 of Procedure 4). Then, within the recursive call on instance  $I_1$ , the algorithm continues either until a contradiction is concluded at a later iteration (in this case a contradiction would be concluded by both  $I_1, I_2^2$ ) or until  $D$  becomes an alternating cycle.

*Correctness for both Cases 1 and 2.* Suppose that, at some iteration of Algorithm 1,  $D$  becomes an alternating cycle, and let  $Q$  be the set of ambivalent edge-quadruples that the algorithm has marked so far. Then the algorithm calls Procedure 2 (see line 8 of the algorithm). Recall by the above analysis of Cases 1 and 2 that, for every ambivalent edge-quadruple  $q \in Q$ , the algorithm had to choose between two alternative edge colorings of the four edges of  $q$ . Furthermore, until the execution of lines 1-3 of Procedure 2, these choices were made arbitrarily, as the choice between these two alternative colorings of  $q$  had no effect on the subsequent iterations of the algorithm. Now, performing the operation Ambivalent-Flip at an ambivalent edge-quadruple  $q \in Q$ , is equivalent to choosing the second alternative coloring of  $q$ .

Assume that, at the beginning of Procedure 2, the symmetric difference  $C_0 \Delta D$  has  $k$  cycles. In lines 1-3 of the procedure, the algorithm attempts to sequentially perform the operation Ambivalent-Flip on all ambivalent quadruples  $q \in Q$ . By flipping the coloring of such a quadruple  $q$ , the number  $k$  of connected components of  $C_0 \Delta D$  can either reduce to  $k - 1$ , or increase to  $k + 1$ , or stay unchanged at  $k$ . Note that, if it decreases to  $k - 1$ , then flipping the colors of  $q$  connects two different connected components (i.e. cycles) of  $C_0 \Delta D$  into one. Similarly, if it increases to  $k + 1$ , then flipping the colors of  $q$  disconnects one cycle of  $C_0 \Delta D$  into two different ones. Finally, if it stays unchanged at  $k$ , then flipping the colors of  $q$  simply replaces one cycle of  $C_0 \Delta D$  with another cycle that visits the same vertices in a different order.

Now note that, performing the operation Ambivalent-Flip at one edge-quadruple  $q$ , does not change the color of an edge in any other quadruple  $q' \in Q \setminus \{q\}$ . Thus, at the end of the execution of lines 2-3 of Procedure 2, the algorithm can decide whether there exists a sequence of choices for the edge-colorings of the ambivalent quadruples in  $Q$  which derive the desired second Hamiltonian cycle  $C_1$ . More specifically, if the symmetric difference  $C_0 \Delta D$  is connected (see line 4 of Procedure 2, where now  $D$  is the updated red-blue cycle after exhaustively executing lines 2-3 of the procedure) then  $C_0 \Delta D$  is the desired second Hamiltonian cycle. Otherwise, if  $C_0 \Delta D$  still contains more than one cycle, it follows that no sequence of choices for the alternative edge-colorings of the quadruples in  $Q$  could derive a second Hamiltonian cycle. This completes the proof of the lemma. ■

**Lemma 3** *Let  $I = (G, C_0, Q, \text{Red}, \text{Blue}, \text{Black}, \text{Yellow})$  be the instance at some iteration of Algorithm 1, where  $G = (V, E)$  is a Smith graph, and let  $D = \text{Red} \cup \text{Blue}$  be the current alternating red-blue path of even length. Then, within a constant number of iterations, either a “contradiction” is announced or the algorithm transforms the instance  $I$  in lines 12-26 either to a single instance  $I'$ , where  $|U(I')| \leq |U(I)| - 2$ , or to two instances  $I_1$  and  $I_2$ , where one of the following is satisfied:*

1.  $|W(I_1)|, |W(I_2)| \leq |W(I)| - 2$  and  $|U(I_1)|, |U(I_2)| \leq |U(I)| - 7$ ,
2.  $|W(I_1)|, |W(I_2)| \leq |W(I)| - 2$  and  $|U(I_1)|, |U(I_2)| \leq |U(I)| - 9$ ,
3.  $|W(I_1)|, |W(I_2)| \leq |W(I)| - 4$  and  $|U(I_1)|, |U(I_2)| \leq |U(I)| - 4$ ,

4.  $|W(I_1)| \leq |W(I)| - 4$ ,  $|U(I_1)| \leq |U(I)| - 4$ , and  $|W(I_2)| \leq |W(I)| - 4$ ,  $|U(I_2)| \leq |U(I)| - 6$ ,
5.  $|W(I_1)| \leq |W(I)| - 2$ ,  $|U(I_1)| \leq |U(I)| - 9$ , and  $|W(I_2)| \leq |W(I)| - 4$ ,  $|U(I_2)| \leq |U(I)| - 6$ ,
6.  $|W(I_1)| \leq |W(I)| - 2$ ,  $|U(I_1)| \leq |U(I)| - 5$ , and  $|W(I_2)| \leq |W(I)| - 4$ ,  $|U(I_2)| \leq |U(I)| - 8$ ,
7.  $|W(I_1)| \leq |W(I)| - 2$ ,  $|U(I_1)| \leq |U(I)| - 3$ , and  $|W(I_2)| \leq |W(I)| - 6$ ,  $|U(I_2)| \leq |U(I)| - 7$ ,
8.  $|W(I_1)| \leq |W(I)| - 2$ ,  $|U(I_1)| \leq |U(I)| - 3$ , and  
 $|W(I_2)| \leq |W(I)| - 4$ ,  $|U(I_2)| \leq |U(I)| - 10$ ,
9.  $|W(I_1)| \leq |W(I)| - 2$ ,  $|U(I_1)| \leq |U(I)| - 3$ , and  $|W(I_2)| \leq |W(I)| - 5$ ,  $|U(I_2)| \leq |U(I)| - 9$ .

**Proof.** Let  $v_y$  be the last vertex of the alternating red-blue path  $D$ , and let  $v_x v_y$  be its last blue chord. Throughout the proof we assume that no “contradiction” is announced at the current iteration. Recall that the set  $W(I)$  contains all biased vertices of instance  $I$ , i.e. all vertices which are incident to an edge of  $Red \cup Black$ . Furthermore recall that the set  $U(I)$  contains all unforced edges of instance  $I$ , i.e. all edges that are not contained in the set  $Red \cup Blue \cup Black \cup Yellow$ . Since  $D$  is an alternating red-blue path (and not a red-blue cycle) by the assumption of the lemma, note that at least one of the two  $C_0$ -edges  $v_y v_{y+1}, v_y v_{y-1}$  is eligible and at most one of them is already colored black, see Observation 4. Assume that one of these two edges is colored and the other one uncolored; note that the colored one can only be black. In this case the other edge is forced to be colored red by the operation Blue-Force. Furthermore, this triggers the operation Red-Force, which forces its uncolored incident chord blue. Thus, in this case the algorithm reduces the problem to a new single instance  $I'$ , which has at least two more edges colored, i.e.  $|U(I')| \leq |U(I)| - 2$ . This corresponds to the case (i) of the lemma.

For the remainder of the proof assume that both  $C_0$ -edges  $v_y v_{y+1}, v_y v_{y-1}$  are eligible. Then the operation Blue-Branch takes place and creates two instances  $I_1, I_2$  (except the cases where line 15 or line 17 of Algorithm 1 is executed, which are dealt with separately in the proof below), where in  $I_1$  the edge  $v_y v_{y+1}$  is red and the edge  $v_y v_{y-1}$  is black, and in  $I_2$  the edge  $v_y v_{y-1}$  is red and the edge  $v_y v_{y+1}$  is black. Note that, in both cases,  $v_y$  becomes a new biased vertex at this iteration as it becomes incident to both a red edge and a black edge. It is not hard to see that the two  $C_0$ -edges  $v_y v_{y+1}, v_y v_{y-1}$  cannot participate together in a forcing cycle. Indeed, in such a forcing cycle  $C$ , one of the edge sequences  $v_y v_{y+1}, v_{y+1} v_{y+1}^*, \dots$  and  $v_y v_{y-1}, v_{y-1} v_{y-1}^*, \dots$  alternatively receives the colors red and blue, while the other one alternatively receives the colors black and yellow. Therefore, there exists exactly one forcing path  $P^-$  starting at vertex  $v_y$  with the edge  $v_y v_{y-1}$ , and exactly one forcing path  $P^+$  starting at vertex  $v_y$  with the edge  $v_y v_{y+1}$ . The next observation follows easily from the fact that for every previously colored edge  $v_i v_j$ , at least one of its endpoints  $v_i, v_j$  is incident to *three* previously colored edges.

**Observation 6** *The two forcing paths  $P^-, P^+$  do not share any common internal vertex.*

*Case 1.* Both  $P^-$  and  $P^+$  end with a  $C_0$ -edge. That is, each of these forcing paths has an even number of *internal* vertices. Since the analysis for both new instances  $I_1, I_2$  is symmetric, in most subcases of Case 1 (with the exception of Case 1(iv)) we only analyze instance  $I_1$ , i.e. the case where  $v_y v_{y+1}$  becomes red and  $v_y v_{y-1}$  becomes black by the operation Blue-Branch. Let  $v_\ell$  be the last vertex of  $P^+$ , and assume without loss of generality that the last edge of  $P^+$  is  $v_{\ell-1} v_\ell$  (the other case, where the last edge of  $P^+$  is  $v_\ell v_{\ell+1}$  is exactly symmetric). Similarly, let  $v_q$  be the last vertex of  $P^-$ , and assume without loss of generality that the last edge of  $P^-$  is  $v_{q-1} v_q$ . That is,  $v_{\ell-1}$  and  $v_{q-1}$  are the last *internal* vertices of  $P^+$  and of  $P^-$ , respectively. Note that  $v_q$  becomes a new biased vertex after the forcing operations along  $P^-$ , as it becomes incident to the new black edge  $v_{q-1} v_q$ . Similarly  $v_\ell$  becomes a new biased vertex after the forcing operations along  $P^+$ , as it becomes incident to the new red edge  $v_{\ell-1} v_\ell$  and to the new black edge  $v_\ell v_{\ell+1}$ . That is,  $v_y, v_q, v_\ell$  become new biased vertices.

*Case 1(i).*  $v_\ell = v_q$ . Then, since  $P^+$  and  $P^-$  share  $v_y$  as a common vertex,  $P^+ \cup P^-$  cannot have just two edges, i.e.  $P^+ \cup P^-$  has at least 4 edges. First assume that  $P^+ \cup P^-$  has 4 edges, that is,  $P^+ \cup P^-$  is a  $C_4$ . Then the algorithm executes line 15 and calls Procedure 3. In this case, it only updates the current



instance  $I$  by forcing the colors of at least 5 edges, namely the 4 edges of  $P^+ \cup P^-$  as well as the chord  $v_\ell v_\ell^*$ . Moreover, the updated instance has at least 2 new biased vertices  $v_y, v_\ell$ . Furthermore the 4 edges of  $P^+ \cup P^-$  are added as an ambivalent quadruple in the set  $Q$  (see line 1 of Procedure 3).

Now assume that  $P^+ \cup P^-$  has exactly 6 edges. Then the algorithm executes lines 18-22 and creates two new instances  $I_1, I_2$ , each of them having at least 2 new biased vertices  $v_y, v_\ell$  and at least 7 new forced edges, namely at least 6 forced edges in  $P^+ \cup P^-$  as well as the blue chord  $v_\ell v_\ell^*$ . Finally, assume that  $P^+ \cup P^-$  has at least 8 edges. Then, similarly to the previous paragraph, we have at least 2 new biased vertices and 9 new forced edges.

Summarizing, in Case 1(i) we either have two instances  $I_1, I_2$ , each having at least 2 new biased vertices and 7 forced edges (when  $P^+ \cup P^-$  has exactly 6 edges), or at least 2 new biased vertices and 9 forced edges (when  $P^+ \cup P^-$  has at least 8 edges), or we just have one updated instance  $I$  which has 2 new biased vertices and 5 forced edges.

*Case 1(ii).*  $v_\ell = v_{q+1}$  (or equivalently,  $v_q = v_{\ell+1}$ ). In this case, when edge  $v_{\ell-1}v_\ell$  (i.e. the last edge of  $P^+$ ) is colored red, the operation Red-Force is triggered which colors the  $C_0$ -edge  $v_\ell v_q$  black and the chord  $v_\ell v_\ell^*$  blue. On the other hand, note that edge  $v_{q-1}v_q$  (i.e. the last edge of  $P^-$ ) is colored black. Thus, since both  $C_0$ -edges  $v_{q-1}v_q$  and  $v_\ell v_q$  are black, the operation Black-Force is triggered at vertex  $v_q$  which colors the chord  $v_q v_q^*$  yellow. Then, the operation Yellow-Force (which is triggered once  $v_q v_q^*$  is colored yellow) colors at least one edge incident to  $v_q^*$  black. Thus, so far we have at the current iteration at least 6 new forced edges, namely at least one forced edge in each of  $P^+$  and  $P^-$ , as well as the edges  $v_\ell v_\ell^*, v_\ell v_q, v_q v_q^*$  and at least one  $C_0$ -edge incident to  $v_q^*$ .

If  $v_q^*$  is incident to two uncolored  $C_0$ -edges then  $v_q^*$  becomes a new biased vertex, i.e. we have four new biased vertices  $v_y, v_q, v_\ell, v_q^*$ . If  $v_q^*$  is incident to two previously colored (i.e. black)  $C_0$ -edges then the edges  $v_q v_q^*$  and  $v_\ell v_q$  have been colored at a previous iteration yellow and black, respectively, which is a contradiction as they received these colors at the current iteration. Finally, if  $v_q^*$  is incident to one previously colored (i.e. black)  $C_0$ -edge and to one uncolored  $C_0$ -edge, then a new forcing path starts at  $v_q$  with the chord  $v_q v_q^*$ . At the end of this forcing path, there must be at least one  $C_0$ -edge incident to an unbiased vertex  $v_z$ , which becomes black after all Black-Force and Yellow-Force operations. Thus  $v_z$  becomes a new biased vertex, that is, we have four new biased vertices  $v_y, v_q, v_\ell, v_z$ .

Summarizing, in Case 1(ii) we have two instances  $I_1, I_2$ , each having at least 4 new biased vertices and 6 forced edges.

*Case 1(iii).*  $v_{\ell+1} = v_{q+1}$ . Similarly to Case 1(ii), when edge  $v_{\ell-1}v_\ell$  is colored red, Red-Force is triggered which colors the  $C_0$ -edge  $v_\ell v_{\ell+1}$  black and the chord  $v_\ell v_\ell^*$  blue. However, since  $v_q$  is the last vertex of  $P^-$ , it follows that the edge  $v_q v_{q+1} = v_q v_{\ell+1}$  is uncolored, and thus  $v_{\ell+1}$  becomes a new biased vertex. That is, we have 4 new biased vertices  $v_y, v_q, v_\ell, v_{\ell+1}$ . Assume that  $P^+ \cup P^-$  has two edges. Then each of  $P^+$  and  $P^-$  has only one edge, namely the  $C_0$ -edges  $v_y v_{y+1}$  and  $v_y v_{y-1}$ , respectively. Then, since the edges  $v_\ell v_{\ell+1}$  and  $v_q v_{\ell+1}$  are also  $C_0$ -edges, it follows that there is a cycle of 4  $C_0$ -edges. This is a contradiction, as  $C_0$  is a Hamiltonian cycle of a graph  $G$  with more than 4 vertices. Thus  $P^+ \cup P^-$  has at least 4 edges, and thus we have at least 6 new forced edges, namely at least 4 forced edges in  $P^+ \cup P^-$ , as well as the edges  $v_\ell v_\ell^*, v_\ell v_{\ell+1}$ .

Summarizing, in Case 1(iii) we have two instances  $I_1, I_2$ , each having at least 4 new biased vertices and 6 forced edges.

*Case 1(iv).*  $\{v_\ell, v_{\ell+1}\} \cap \{v_q, v_{q+1}\} = \emptyset$ . Similarly to Cases 1(ii) and 1(iii), the chord  $v_\ell v_\ell^*$  is colored blue and the  $C_0$ -edge  $v_\ell v_{\ell+1}$  is colored black by the Red-Force operation triggered at the end of  $P^+$ . If  $v_{\ell+1}v_{\ell+2}$  is uncolored then vertex  $v_{\ell+1}$  is a new biased vertex, that is, we have 4 new biased vertices  $v_y, v_q, v_\ell, v_{\ell+1}$ . Now assume that  $v_{\ell+1}v_{\ell+2}$  is a previously colored (i.e. black)  $C_0$ -edge. Then, at the end of  $P^+$  we have a new forcing path starting at vertex  $v_\ell$  with the edge  $v_\ell v_{\ell+1}$ . At the end of this forcing path, there must be at least one  $C_0$ -edge incident to an unbiased vertex  $v_z$ , which becomes black after all Black-Force and Yellow-Force operations. Note that vertex  $v_q$  can possibly be one of the internal vertices of this new forcing path. Thus  $v_z$  becomes a new biased vertex, that is, we have 4 new biased vertices  $v_y, v_q, v_\ell, v_z$ .

Suppose that *both* edges  $v_{\ell+1}v_{\ell+2}$  and  $v_{q+1}v_{q+2}$  are not previously colored black. Then, in both instances  $I_1, I_2$  we have at least 4 new forced edges, namely at least one forced edge in each of  $P^+$  and  $P^-$ , as well

as the edges  $v_\ell v_\ell^*, v_\ell v_{\ell+1}$  for  $I_1$  (resp. the edges  $v_q v_q^*, v_q v_{q+1}$  for  $I_2$ ). Note here this case is only possible to appear when, until the current iteration, vertex  $v_y$  is in the center of a path of at least 6 consecutive  $C_0$ -edges that have not been colored yet, namely the edges  $v_{y-3}v_{y-2}, v_{y-2}v_{y-1}, v_{y-1}v_y, v_y v_{y+1}, v_{y+1}v_{y+2}, v_{y+2}v_{y+3}$ .

Now suppose that at least one of the edges  $v_{\ell+1}v_{\ell+2}$  and  $v_{q+1}v_{q+2}$  is a previously colored black edge, say this edge is  $v_{\ell+1}v_{\ell+2}$  without loss of generality. Then, although in the instance  $I_2$  we have again at least 4 new forced edges, in the instance  $I_1$  we additionally have the chord  $v_{\ell+1}v_{\ell+1}^*$  and at least one  $C_0$ -edge incident to  $v_{\ell+1}^*$  which are colored yellow and black, respectively. That is,  $I_1$  has in this case at least 6 new forced edges.

Summarizing, in Case 1(iv) we have two instances  $I_1, I_2$ , either each having at least 4 new biased vertices and 4 forced edges, or one instance having at least 4 new biased vertices and 4 forced edges and the other instance having at least 4 new biased vertices and 6 forced edges.

*Case 2.* Both  $P^-$  and  $P^+$  end with a chord. That is, each of these forcing paths has an odd number of *internal* vertices. Since the analysis for both new instances  $I_1, I_2$  is symmetric, here we only analyze instance  $I_1$ , i.e. the case where  $v_y v_{y+1}$  becomes red and  $v_y v_{y-1}$  becomes black by the operation Blue-Branch. Let  $v_\ell$  be the last vertex of  $P^+$  and let the chord  $v_\ell^* v_\ell$  be the last edge of  $P^+$ . Similarly, let  $v_q$  be the last vertex of  $P^-$  and let the chord  $v_q^* v_q$  be the last edge of  $P^-$ . That is,  $v_\ell^*$  and  $v_q^*$  are the last *internal* vertices of  $P^+$  and of  $P^-$ , respectively. Since  $P^+$  and  $P^-$  share no common internal vertices by Observation 6, it follows that  $v_\ell^* \neq v_q^*$ , and thus also  $v_\ell \neq v_q$  (as every vertex is incident to exactly one chord). Note that  $v_q$  becomes a new biased vertex after the forcing operations along  $P^-$ , as it becomes incident to the new black edges  $v_q v_{q-1}$  and  $v_q v_{q+1}$ . That is,  $v_y$  and  $v_q$  become new biased vertices.

*Case 2(i).*  $v_\ell \in \{v_{q-1}, v_{q+1}\}$ , i.e.  $v_\ell v_q$  is one of the two  $C_0$ -edges that are incident to  $v_q$ . Assume without loss of generality that  $v_\ell = v_{q-1}$ , or equivalently  $v_q = v_{\ell+1}$ . Then the  $C_0$ -edges incident to  $v_\ell$  are  $v_\ell v_q$  and  $v_\ell v_{\ell-1}$ . In this case the operation Yellow-Force which is triggered at the end of  $P^-$  (once the chord  $v_q^* v_q$  is colored yellow) colors both  $C_0$ -edges  $v_\ell v_q$  and  $v_q v_{q+1}$  black. Thus, since  $v_\ell v_q$  becomes black and  $v_\ell v_\ell^*$  becomes blue, the operation Blue-Force colors the  $C_0$ -edge  $v_\ell v_{\ell-1}$  red. Furthermore, the operation Red-Force colors the chord  $v_{\ell-1} v_{\ell-1}^*$  blue. That is, we have at least 8 new forced edges, i.e. at least two forced edges in each of  $P^+, P^-$ , as well as the edges  $v_\ell v_q, v_q v_{q+1}, v_\ell v_{\ell-1}, v_{\ell-1} v_{\ell-1}^*$ .

Furthermore,  $v_\ell$  becomes a new biased vertex as it becomes incident to a new black edge and a new red edge. If  $v_{q+1}$  is not incident to any previously colored black  $C_0$ -edge, then  $v_{q+1}$  becomes a new biased vertex, that is, we have at least 4 new biased vertices  $v_y, v_q, v_\ell, v_{q+1}$ . Otherwise, if  $v_{q+1}$  is incident to a previously colored black  $C_0$ -edge, then a new forcing path starts at  $v_{q+1}$  with the edge  $v_{q+1} v_{q+1}^*$ . At the end of this forcing path, there must be at least one  $C_0$ -edge incident to an unbiased vertex  $v_z$ , which becomes black after all Black-Force and Yellow-Force operations. Thus  $v_z$  becomes a new biased vertex, that is, we have at least 4 new biased vertices  $v_y, v_q, v_\ell, v_z$ .

Summarizing, in Case 2(i) we have two instances  $I_1, I_2$ , each having at least 4 new biased vertices and 8 forced edges.

*Case 2(ii).*  $v_q \in \{v_{\ell-1}, v_{\ell+1}\}$ . This case is equivalent to Case 2(i).

*Case 2(iii).*  $\{v_\ell, v_{\ell-1}, v_{\ell+1}\} \cap \{v_q, v_{q-1}, v_{q+1}\} = \emptyset$ . Assume that both  $v_{q-1}, v_{q+1}$  are not incident to any previously colored black vertices. Then both  $v_{q-1}, v_{q+1}$  become new biased vertices, i.e. we have in total 4 new biased vertices  $v_y, v_q, v_{q-1}, v_{q+1}$ . Furthermore we have at least 6 new forced edges, namely at least two forced edges in each of  $P^+$  and  $P^-$ , as well as the edges  $v_q v_{q-1}, v_q v_{q+1}$ .

Now assume that one of the vertices  $v_{q-1}, v_{q+1}$  (say  $v_{q-1}$ ) is incident to a previously colored black vertex and the other one (say  $v_{q+1}$ ) is not. Then, on the one hand,  $v_{q+1}$  becomes a new biased vertex. On the other hand, a new forcing path  $P^\#$  starts at vertex  $v_q$  with the edge  $v_q v_{q-1}$ . At the end of this forcing path  $P^\#$ , there must be at least one  $C_0$ -edge incident to an unbiased vertex  $v_z$ , which becomes black after all Black-Force and Yellow-Force operations. Note that vertex  $v_{q+1}$  can possibly be one of the internal vertices of this new forcing path  $P^\#$ . Thus  $v_z$  becomes a new biased vertex, that is, we have 4 new biased vertices  $v_y, v_q, v_{q+1}, v_z$ . Furthermore we have in this case at least 8 new forced edges, namely at least two forced edges in each of  $P^+$  and  $P^-$ , the edges  $v_q v_{q-1}, v_q v_{q+1}$ , and at least two more new forced edges (at least one chord and one  $C_0$ -edge) in the new forcing path  $P^\#$ .

Next assume that each of the vertices  $v_{q-1}, v_{q+1}$  is incident to a previously colored black vertex. Then,

at vertex  $v_q$  we have either two new forcing paths (one starting with the edge  $v_q v_{q-1}$  and one starting with the edge  $v_q v_{q+1}$ ) or one forcing cycle (containing the edges  $v_q v_{q-1}$  and  $v_q v_{q+1}$ ). If we have one forcing cycle, it must contain at least 5 new forced edges (as  $G$  is without loss of generality triangle-free by Theorem 1). This implies that we have in total at least two new biased vertices  $v_y, v_q$  and at least 9 new forced edges, namely at least two forced edges in each of  $P^+$  and  $P^-$ , as well as at least 5 more forced edges in the forcing cycle at  $v_q$ .

Finally assume that we have two forcing paths  $P^\#$  and  $P^*$ , starting with the edge  $v_q v_{q-1}$  and with the edge  $v_q v_{q+1}$ , respectively. At the end of each of these forcing paths  $P^\#$  and  $P^*$ , there must be at least one  $C_0$ -edge incident to an unbiased vertex  $v_{z^\#}$  and  $v_{z^*}$ , respectively, which becomes black after all Black-Force and Yellow-Force operations. Thus each of the vertices  $v_{z^\#}$  and  $v_{z^*}$  becomes a new biased vertex. Assume that  $v_{z^\#} \neq v_{z^*}$ . Then we have 4 new biased vertices  $v_y, v_q, v_{z^\#}, v_{z^*}$ . Furthermore we have at least 10 new forced edges, namely at least two forced edges in each of  $P^+$  and  $P^-$ , the edges  $v_q v_{q-1}, v_q v_{q+1}$ , and at least two more new forced edges (at least one  $C_0$ -edge and one chord) in each of the new forcing paths  $P^\#$  and  $P^*$ . Now assume that  $v_{z^\#} = v_{z^*}$ . Then there exists a new forcing path  $P^{**}$  starting at  $v_{z^\#}$ . At the end of this forcing path  $P^{**}$  there must be at least one  $C_0$ -edge incident to an unbiased vertex  $v_z$ , which becomes black after all Black-Force and Yellow-Force operations. Thus  $v_z$  becomes a new biased vertex, that is, we have 4 new biased vertices  $v_y, v_q, v_{z^\#}, v_z$ . Furthermore, we have in this case at least 10 new forced edges, namely at least two forced edges in each of  $P^+$  and  $P^-$ , at least 5 forced edges in  $P^\# \cup P^*$ , and at least one more forced edge in  $P^{**}$ .

Summarizing, in Case 2(iii) we have two instances  $I_1, I_2$ , each of them having either at least 2 new biased vertices and 9 forced edges, or at least 4 new biased vertices and 6 forced edges.

*Case 3.*  $P^-$  ends with a chord and  $P^+$  ends with a  $C_0$ -edge. That is,  $P^-$  has an odd number and  $P^+$  has an even number of *internal* vertices. Here the analysis for the two new instances  $I_1, I_2$  is not symmetric, so we will analyze them separately.

*Case 3(i).*  $P^+ \cup P^-$  builds a  $P_4$  (i.e. a path with 4 vertices) whose two endpoints are adjacent. Then the algorithm executes line 17 and calls Procedure 4. As  $P^+ \cup P^-$  builds a  $P_4$ , either  $P^+$  contains one edge and  $P^-$  contains two edges (see lines 2-4 of Procedure 4), or  $P^+$  contains two edges and  $P^-$  contains one edge (see lines 5-7 of Procedure 4). As the analysis of both these cases is symmetric, we only consider here the first case, i.e. that  $P^+$  contains one edge and  $P^-$  contains two edges. In this case the algorithm branches to two new instances  $I_1, I_2$ , as follows. In  $I_1$ , the  $C_0$ -edge  $v_y v_{y+1}$  becomes red, the  $C_0$ -edges  $v_y v_{y-1}, v_{y+1} v_{y+2}, v_{y+2} v_{y+3}$  become black, the chord  $v_{y+1} v_{y+1}^*$  becomes blue, and the chord  $v_{y-1} v_{y+2}$  becomes yellow. Furthermore the 4 edges  $v_y v_{y+1}, v_y v_{y-1}, v_{y-1} v_{y+2}, v_{y+2} v_{y+1}$  are added as an ambivalent quadruple in the set  $Q$  (see line 3 of Procedure 4). In  $I_2$ , the  $C_0$ -edges  $v_y v_{y-1}, v_{y+2} v_{y+3}$  become red, the  $C_0$ -edges  $v_y v_{y+1}, v_{y+1} v_{y+2}$  become black, the chords  $v_{y-1} v_{y+2}, v_{y+3} v_{y+1}^*$  become blue, and the chord  $v_{y+1} v_{y+1}^*$  becomes yellow. Furthermore, at least one more (previously uncolored) incident  $C_0$ -edge of  $v_{y+1}^*$  becomes black by the Yellow-Force operation that is triggered once  $v_{y+1} v_{y+1}^*$  becomes yellow. Thus,  $I_1$  has at least 6 new forced edges and  $I_2$  has at least 8 new forced edges.

In  $I_1$ , if  $v_{y+3}$  is not incident to any previously colored black edge, it becomes a new biased vertex. That is, in this case we have in  $I_1$  in total at least 4 new biased vertices  $v_y, v_{y+1}, v_{y+2}, v_{y+3}$ . Otherwise, if  $v_{y+3}$  is incident to a previously colored black edge, a new forcing path starts at  $v_{y+3}$ . At the end of this forcing path there must be at least one  $C_0$ -edge incident to an unbiased vertex  $v_z$ , which becomes black after all Black-Force and Yellow-Force operations. Thus  $v_z$  becomes a new biased vertex, that is, we have 4 new biased vertices  $v_y, v_{y+1}, v_{y+2}, v_z$ .

In  $I_2$ , if  $v_{y+1}^*$  is not incident to any previously colored black edge, it becomes a new biased vertex. That is, in this case we have in  $I_2$  in total at least 4 new biased vertices  $v_y, v_{y+1}, v_{y+2}, v_{y+1}^*$ . Otherwise, if  $v_{y+1}^*$  is incident to a previously colored black edge, a new forcing path starts at  $v_{y+1}^*$ . At the end of this forcing path there must be at least one  $C_0$ -edge incident to an unbiased vertex  $v_z$ , which becomes black after all Black-Force and Yellow-Force operations. Thus  $v_z$  becomes a new biased vertex, that is, we have 4 new biased vertices  $v_y, v_{y+1}, v_{y+2}, v_z$ .

Summarizing, in Case 3(i),  $I_1$  has at least 4 new biased vertices and 6 new forced edges, while  $I_2$  has at least 4 new biased vertices and 8 new forced edges.

*Case 3(ii).*  $P^+ \cup P^-$  does not build a  $P_4$  whose two endpoints are adjacent. Since  $P^-$  ends with a chord and  $P^+$  ends with a  $C_0$ -edge by the assumption of Case 3, it easily follows that  $P^+ \cup P^-$  does also not build a  $C_4$ . Therefore Algorithm 1 executes 18-22 and branches to two new instances  $I_1, I_2$ , which we analyze separately below. Let  $v_\ell$  be the last vertex of  $P^+$ , and assume without loss of generality that the last edge of  $P^+$  is  $v_{\ell-1}v_\ell$  (the other case, where the last edge of  $P^+$  is  $v_\ell v_{\ell+1}$  is exactly symmetric). Furthermore, let  $v_q$  be the last vertex of  $P^-$  and let the chord  $v_q^*v_q$  be the last edge of  $P^-$ . That is,  $v_{\ell-1}$  and  $v_q^*$  are the last internal vertices of  $P^+$  and of  $P^-$ , respectively.

*Case 3(ii)(a).*  $v_\ell \in \{v_{q-1}, v_{q+1}\}$ , i.e.  $v_\ell v_q$  is one of the two  $C_0$ -edges that are incident to  $v_q$ . Assume without loss of generality that  $v_\ell = v_{q-1}$ , or equivalently  $v_q = v_{\ell+1}$ . Then the  $C_0$ -edges incident to  $v_\ell$  are  $v_\ell v_q$  and  $v_\ell v_{\ell-1}$ . Note that, if  $P^+ \cup P^-$  has three edges, then  $P^+ \cup P^-$  builds a  $P_4$  whose two endpoints are adjacent, which is a contradiction to the assumption of Case 3(ii). Thus  $P^+ \cup P^-$  has at least 5 edges.

In the instance  $I_1$ , the operation Yellow-Force which is triggered at the end of  $P^-$  (once the chord  $v_q^*v_q$  is colored yellow) colors both  $C_0$ -edges  $v_\ell v_q$  and  $v_q v_{q+1}$  black. Furthermore, when the  $C_0$ -edge  $v_{\ell-1}v_\ell$  becomes red, the operation Red-Force colors the chord  $v_\ell v_\ell^*$  blue. That is, we have at least 8 new forced edges, namely at least 5 forced edges in  $P^+ \cup P^-$ , as well as the edges  $v_\ell v_\ell^*, v_\ell v_q, v_q v_{q+1}$ . If  $v_{q+1}v_{q+2}$  is not a black edge, then  $v_{q+1}$  becomes a new biased vertex, that is, we have the 4 biased vertices  $v_y, v_\ell, v_q, v_{q+1}$ . Otherwise, if  $v_{q+1}v_{q+2}$  is a black edge, then a new forcing path starts at  $v_{q+1}$  with the chord  $v_{q+1}v_{q+1}^*$ . At the end of this forcing path, there must be at least one  $C_0$ -edge incident to an unbiased vertex  $v_z$ , which becomes black after all Black-Force and Yellow-Force operations. Thus  $v_z$  becomes a new biased vertex, that is, we have 4 new biased vertices  $v_y, v_\ell, v_q, v_z$ .

In the instance  $I_2$ , recall that  $v_y v_{y+1}$  becomes black and  $v_y v_{y-1}$  becomes red by the operation Blue-Branch. Note that here  $v_\ell$  becomes a new biased vertex after the forcing operations along  $P^+$ , as it becomes incident to the new black edge  $v_{\ell-1}v_\ell$ . That is, we have at least the 2 new biased vertices  $v_y, v_\ell$ . Furthermore we have at least 5 new forced edges, namely the edges  $P^+ \cup P^-$ .

Summarizing, in Case 3(ii)(a),  $I_1$  has at least 4 new biased vertices and 8 forced edges, while  $I_2$  has at least 2 new biased vertices and 5 forced edges.

*Case 3(ii)(b).*  $v_\ell \notin \{v_{q-1}, v_{q+1}\}$ . In the instance  $I_2$  (i.e. where the  $C_0$ -edges  $v_y v_{y+1}$  and  $v_y v_{y-1}$  become black and red, respectively)  $v_\ell$  becomes a new biased vertex after the forcing operations along  $P^+$ , as it becomes incident to the new black edge  $v_{\ell-1}v_\ell$ . That is, we have at least the 2 new biased vertices  $v_y, v_\ell$ . Furthermore we have at least 3 new forced edges, namely at least one forced edge in  $P^+$  and at least two forced edges in  $P^-$ . That is, in Case 3(ii)(b),  $I_2$  has at least 2 new biased vertices and 3 forced edges.

*Analysis of instance  $I_1$ .* Recall that here  $v_y v_{y+1}$  becomes red and that  $v_y v_{y-1}$  becomes black by the operation Blue-Branch. Note that  $v_q$  becomes a new biased vertex after the forcing operations along  $P^-$ , as it becomes incident to the new black edges  $v_q v_{q-1}$  and  $v_q v_{q+1}$ . Moreover, note that  $v_\ell$  becomes a new biased vertex after the forcing operations along  $P^+$ , as it becomes incident to the new red edge  $v_{\ell-1}v_\ell$  and to the new black edge  $v_\ell v_{\ell+1}$ . That is,  $v_y, v_\ell, v_q$  become new biased vertices.

*Case 3(ii)(b)(1).*  $v_\ell = v_q$ . In this case, after the forcing operations along  $P^-$ , vertex  $v_\ell$  is incident to the two black  $C_0$ -edges  $v_q v_{q-1}, v_q v_{q+1}$ , and thus the edge  $v_{\ell-1}v_\ell$  cannot be colored red in the forcing path  $P^+$ , which is a contradiction.

*Case 3(ii)(b)(2).*  $v_{\ell+1} \in \{v_{q-1}, v_{q+1}\}$ . Then, since the case  $v_{\ell+1} = v_{q+1}$  is equivalent to the case  $v_\ell = v_q$  (which has been dealt with in Case 3( $I_1$ )(i)), we assume that  $v_{\ell+1} = v_{q-1}$ . The Red-Force operation, which is triggered when the last edge  $v_{\ell-1}v_\ell$  of  $P^+$  becomes red, colors the  $C_0$ -edge  $v_\ell v_{\ell+1}$  black and the chord  $v_\ell v_\ell^*$  blue. Furthermore the Yellow-Force operation, which is triggered when the last edge  $v_q^*v_q$  of  $P^-$  becomes yellow, colors both  $C_0$ -edges  $v_q v_{q-1} = v_q v_{\ell+1}$  and  $v_q v_{q+1}$  black. Thus vertex  $v_{\ell+1}$  becomes a new biased vertex. That is, we have at least 4 new biased vertices  $v_y, v_\ell, v_q, v_{\ell+1}$ .

Assume that  $v_{q+1}$  is incident to a previously colored black edge. Then, once  $v_q v_{q+1}$  is colored black, the operation Black-Force is triggered which colors the chord  $v_{q+1}v_{q+1}^*$  yellow. On the other hand, once  $v_\ell v_{\ell+1}$  and  $v_q v_{\ell+1}$  are colored black, the Black-Force and Yellow-Force operations are triggered, and thus the chord  $v_{\ell+1}v_{\ell+1}^*$  becomes yellow and at least one of the two  $C_0$ -edges incident to  $v_{\ell+1}^*$  becomes black. Thus we have at least 10 new forced edges, namely at least one forced edge in  $P^+$ , at least two forced edges in  $P^-$ , the edges  $v_\ell v_\ell^*, v_\ell v_{\ell+1}, v_q v_{\ell+1}, v_q v_{q+1}, v_{q+1}v_{q+1}^*, v_{\ell+1}v_{\ell+1}^*$ , as well as at least one  $C_0$ -edge incident to  $v_{\ell+1}^*$ .

Now assume that  $v_{q+1}$  is not incident to any previously colored black edge. Then, once  $v_\ell v_{\ell+1}$  and  $v_q v_{\ell+1}$  are colored black, the Black-Force operation is triggered, which colors the chord  $v_{\ell+1} v_{\ell+1}^*$  yellow. Let  $v_t = v_{\ell+1}^*$ , i.e. the two  $C_0$ -edges incident to  $v_{\ell+1}^*$  are  $v_{\ell+1}^* v_{t-1}$  and  $v_{\ell+1}^* v_{t+1}$ . Note that at least one of these edges  $v_{\ell+1}^* v_{t-1}$  and  $v_{\ell+1}^* v_{t+1}$  is uncolored, as otherwise  $v_{\ell+1} v_{\ell+1}^*$  would have been colored yellow at a previous iteration, which is a contradiction. Furthermore note that both vertices  $v_{t-1}$  and  $v_{t+1}$  are different than  $v_q$ , since otherwise  $G$  would have a triangle on the vertices  $v_{\ell+1}, v_{\ell+1}^*, v_q$ , which is a contradiction by Theorem 1.

Suppose that  $v_{q+1} \in \{v_{t-1}, v_{t+1}\}$ , say without loss of generality that  $v_{q+1} = v_{t-1}$ . Then the edge  $v_{\ell+1}^* v_{t-1} = v_{\ell+1}^* v_{q+1}$  is currently uncolored, as we assumed that  $v_{q+1}$  is not incident to any previously colored black edge. Once  $v_{\ell+1} v_{\ell+1}^*$  is colored yellow, the operation Yellow-Force is triggered which colors the edge  $v_{\ell+1}^* v_{q+1}$  black. Thus, as both  $v_{\ell+1}^* v_{q+1}$  and  $v_q v_{q+1}$  are colored black, the Black-Force operation is triggered which colors the chord  $v_{q+1} v_{q+1}^*$  yellow. Thus we have again at least 10 new forced edges, namely at least one forced edge in  $P^+$ , at least two forced edges in  $P^-$ , and the edges  $v_\ell v_\ell^*, v_\ell v_{\ell+1}, v_q v_{\ell+1}, v_q v_{q+1}, v_{q+1} v_{q+1}^*, v_{\ell+1} v_{\ell+1}^*, v_{\ell+1}^* v_{q+1}$ .

Finally suppose that  $v_{q+1} \notin \{v_{t-1}, v_{t+1}\}$ . If both edges  $v_{\ell+1}^* v_{t-1}$  and  $v_{\ell+1}^* v_{t+1}$  are uncolored, they both become black by the Yellow-Force operation that is triggered at  $v_{\ell+1}^*$ , once  $v_{\ell+1} v_{\ell+1}^*$  becomes yellow. Thus in this case we have at least 10 new forced edges, namely at least one forced edge in  $P^+$ , at least two forced edges in  $P^-$ , and the edges  $v_\ell v_\ell^*, v_\ell v_{\ell+1}, v_q v_{\ell+1}, v_q v_{q+1}, v_{\ell+1} v_{\ell+1}^*, v_{\ell+1}^* v_{t-1}, v_{\ell+1}^* v_{t+1}$ . Otherwise, let one of the edges  $v_{\ell+1}^* v_{t-1}$  and  $v_{\ell+1}^* v_{t+1}$  (say, the edge  $v_{\ell+1}^* v_{t-1}$ ) be previously colored black. Then we have at least 9 new forced edges, namely at least one forced edge in  $P^+$ , at least two forced edges in  $P^-$ , and the edges  $v_\ell v_\ell^*, v_\ell v_{\ell+1}, v_q v_{\ell+1}, v_q v_{q+1}, v_{\ell+1} v_{\ell+1}^*, v_{\ell+1}^* v_{t+1}$ . If the  $C_0$ -edge  $v_{t+1} v_{t+2}$  is uncolored, then  $v_{t+1}$  is a new biased vertex, and thus we have at least 5 new biased vertices  $v_y, v_\ell, v_q, v_{\ell+1}, v_{t+1}$ . Otherwise, if  $v_{t+1} v_{t+2}$  is black, then a new forcing path starts at  $v_{\ell+1}^*$  with the edge  $v_{\ell+1}^* v_{t+1}$ . At the end of this forcing path, there must be at least one  $C_0$ -edge incident to an unbiased vertex  $v_z$ , which becomes black after all Black-Force and Yellow-Force operations. Thus  $v_z$  becomes a new biased vertex, that is, we have again at least 5 new biased vertices  $v_y, v_\ell, v_q, v_{\ell+1}, v_z$ .

Summarizing, in Case 3(ii)(b)(2),  $I_1$  has either at least 4 new biased vertices and 10 new forced edges, or at least 5 new biased vertices and 9 new forced edges.

*Case 3(ii)(b)(3).*  $v_{\ell+1} = v_q$ . This case is equivalent to Case 3(ii)(a).

*Case 3(ii)(b)(4).*  $\{v_\ell, v_{\ell+1}\} \cap \{v_q, v_{q-1}, v_{q+1}\} = \emptyset$ . First suppose that none of the three vertices  $v_{\ell+1}, v_{q-1}, v_{q+1}$  is incident to any previously colored black edge. Then all these three vertices become new biased vertices, and thus we have in total at least 6 new biased vertices (i.e. together with  $v_y, v_\ell, v_q$ ). Furthermore, in this case we have at least 7 new forced edges, namely at least one forced edge in  $P^+$ , at least two forced edges in  $P^-$ , as well as the four edges  $v_\ell v_\ell^*, v_\ell v_{\ell+1}, v_q v_{q-1}, v_q v_{q+1}$ .

Suppose that exactly one of the three vertices  $v_{\ell+1}, v_{q-1}, v_{q+1}$  is incident to a previously colored black edge; denote this vertex by  $v_t$ . Then, the two vertices in  $\{v_{\ell+1}, v_{q-1}, v_{q+1}\} \setminus \{v_t\}$  are new biased vertices, i.e. we have in total at least 5 new biased vertices (together with  $v_y, v_\ell, v_q$ ). Furthermore, the Black-Force operation is triggered at  $v_t$ , which colors the chord  $v_t v_t^*$  yellow. Moreover the Yellow-Force operation is forced at  $v_t^*$ , which colors at least one of the  $C_0$ -edges incident to  $v_t^*$  black. Thus we have at least 9 new forced edges, namely at least one forced edge in  $P^+$ , at least two forced edges in  $P^-$ , as well as the edges  $v_\ell v_\ell^*, v_\ell v_{\ell+1}, v_q v_{q-1}, v_q v_{q+1}, v_t v_t^*$  and at least one of the  $C_0$ -edges incident to  $v_t^*$ .

Now suppose that exactly two of the three vertices  $v_{\ell+1}, v_{q-1}, v_{q+1}$  are incident to a previously colored black edge; denote these vertices by  $v_{t_1}, v_{t_2}$ . Then, the single vertex in  $\{v_{\ell+1}, v_{q-1}, v_{q+1}\} \setminus \{v_{t_1} v_{t_2}\}$  is a new biased vertex, i.e. we have in total at least 4 new biased vertices (together with  $v_y, v_\ell, v_q$ ). Furthermore, the Black-Force operation is triggered both at  $v_{t_1}$  and at  $v_{t_2}$ , which color the chords  $v_{t_1} v_{t_1}^*$  and  $v_{t_2} v_{t_2}^*$  yellow. Moreover, the Yellow-Force operations that are forced at  $v_{t_1}^*$  and  $v_{t_2}^*$  colors at least one of the  $C_0$ -edges incident to  $v_{t_1}^*$  or  $v_{t_2}^*$  black (note that this  $C_0$ -edge may be incident to both  $v_{t_1}^*$  and  $v_{t_2}^*$ ). Thus we have at least 10 new forced edges, namely at least one forced edge in  $P^+$ , at least two forced edges in  $P^-$ , as well as the edges  $v_\ell v_\ell^*, v_\ell v_{\ell+1}, v_q v_{q-1}, v_q v_{q+1}, v_{t_1} v_{t_1}^*, v_{t_2} v_{t_2}^*$  and at least one of the  $C_0$ -edges incident to  $v_{t_1}^*$  or  $v_{t_2}^*$ .

Finally suppose that all three vertices  $v_{\ell+1}, v_{q-1}, v_{q+1}$  are incident to a previously colored black edge. Then the Black-Force operations that are triggered at these three vertices which color the

chords  $v_{\ell+1}v_{\ell+1}^*$ ,  $v_{q-1}v_{q-1}^*$ , and  $v_{q+1}v_{q+1}^*$  yellow. Thus we have at least 10 new forced edges, namely at least one forced edge in  $P^+$ , at least two forced edges in  $P^-$ , as well as the edges  $v_\ell v_\ell^*$ ,  $v_\ell v_{\ell+1}$ ,  $v_q v_{q-1}$ ,  $v_q v_{q+1}$ ,  $v_{\ell+1}v_{\ell+1}^*$ ,  $v_{q-1}v_{q-1}^*$ ,  $v_{q+1}v_{q+1}^*$ . Furthermore, at the end of at least one of the new forcing paths starting at the vertices  $v_{\ell+1}$ ,  $v_{q-1}$ , and  $v_{q+1}$  with the edges  $v_{\ell+1}v_{\ell+1}^*$ ,  $v_{q-1}v_{q-1}^*$ , and  $v_{q+1}v_{q+1}^*$ , respectively, there must be at least one  $C_0$ -edge incident to an unbiased vertex  $v_z$ , which becomes black after all Black-Force and Yellow-Force operations. Thus  $v_z$  becomes a new biased vertex, that is, we have again at least 4 new biased vertices  $v_y, v_\ell, v_q, v_z$ .

Summarizing, in Case 3(ii)(b)(4),  $I_1$  has either at least 6 new biased vertices and 7 new forced edges, or at least 5 new biased vertices and 9 new forced edges, or at least 4 new biased vertices and 10 new forced edges. ■

We are now ready to use the results of our technical Lemma 3 to derive an upper bound for the running time of Algorithm 1.

**Theorem 3** *Let  $G$  be a Smith graph on  $n$  vertices with a given Hamiltonian cycle  $C_0$ . If  $G$  does not contain any induced cycle  $C_6$  on 6 vertices, then the running time of Algorithm 1 is  $O(n \cdot 2^{0.2971925n})$ . Otherwise the running time is  $O(n \cdot 2^{(0.3-\varepsilon)n})$ , for some constant  $\varepsilon > 0$ .*

**Proof.** We derive the running time of the algorithm from the inequalities given in the statement of Lemma 3. These inequalities upper bound the sizes of the sets  $W(I_1), W(I_2)$  of unbiased vertices and of the sets  $U(I_1), U(I_2)$  of unforced edges in the two instances  $I_1, I_2$  which are obtained in the various cases where the algorithm branches. In fact, for upper-bounding the running time, we can ignore the cases where the algorithm just updates the current instance  $I$  (instead of branching to two new instances  $I_1, I_2$ ), as in these cases the current instance shrinks in a constant number of steps by Lemma 3.

Given an instance  $I$  at some iteration of the algorithm, where  $|W(I)| = x$  and  $|U(I)| = y$ , we denote by  $f(x, y)$  the worst-case running time needed for the algorithm to compute all the desired alternating red-blue cycles  $D = \text{Red} \cup \text{Blue}$  of  $I$ , (or to announce “contradiction” in the branches that such a cycle  $D$  does not exist). Recall that, with the help of the ambivalent quadruples, these alternating cycles  $D$  computed by the algorithm form a succinct encoding of all possible alternating red-blue cycles.

In any of the inequalities of Lemma 3, whenever the instance  $I$  is replaced by the instances  $I_1, I_2$  and the values  $x, y$  in  $I$  are replaced by the values  $x - k_1, y - \ell_1$  and by  $x - k_2, y - \ell_2$  in  $I_1$  and  $I_2$ , respectively, then the running time  $f(x, y)$  of the algorithm at  $I$  is replaced by the sum  $f(x - k_1, y - \ell_1) + f(x - k_2, y - \ell_2)$  of the running times at  $I_1$  and  $I_2$ , respectively. That is, in the worst case,  $f(x, y) \leq f(x - k_1, y - \ell_1) + f(x - k_2, y - \ell_2)$ . However, in order to compute an upper bound for  $f(x, y)$  from all the recurrences, we need to substitute “ $\leq$ ” by “ $\geq$ ” in each inequality. Thus we obtain the following system of recurrence inequalities immediately by the statement of Lemma 3.

1.  $f(x, y) \geq 2f(x - 2, y - 7)$ ,
2.  $f(x, y) \geq 2f(x - 4, y - 4)$ ,
3.  $f(x, y) \geq 2f(x - 2, y - 9)$ ,
4.  $f(x, y) \geq f(x - 4, y - 4) + f(x - 4, y - 6)$ ,
5.  $f(x, y) \geq f(x - 2, y - 9) + f(x - 4, y - 6)$ ,
6.  $f(x, y) \geq f(x - 2, y - 5) + f(x - 4, y - 8)$ ,
7.  $f(x, y) \geq f(x - 2, y - 3) + f(x - 6, y - 7)$ ,
8.  $f(x, y) \geq f(x - 2, y - 3) + f(x - 4, y - 10)$ ,
9.  $f(x, y) \geq f(x - 2, y - 3) + f(x - 5, y - 9)$ .

For the sake of presentation, we divide the analysis of the above recurrences into three parts. In Part A we give a naive upper bound on the solution of the above recurrences in worst case. In Part B we give an upper bound in the case where the input Smith graph  $G$  does not contain any induced cycle  $C_6$  on 6 vertices. In Part C we present a more sophisticated analysis of the worst case (including  $C_6$ 's), thus obtaining an improved upper bound.

*Part A: A naive upper bound of the worst case.* To solve this system of recurrences, we set  $f(x, y) = 2^{\alpha x + \beta y}$  and we compute the optimum values for  $\alpha, \beta$  that satisfy all the above inequalities. As it can be easily verified, the values  $\{\alpha = 0.15, \beta = 0.1\}$  satisfy all these inequalities together. Moreover it can be easily checked that, with these values of  $\alpha, \beta$ , the first two relations become *equalities*, while all the other relations become *strict inequalities*. Since  $x \leq n$  and  $y \leq \frac{3}{2}n$ , it follows that  $\alpha x + \beta y \leq (\alpha + \frac{3}{2}\beta)n = 0.3 \cdot n$ , and thus the running time of the algorithm in the worst case is upper-bounded by  $2^{0.3 \cdot n}$ .

*Part B: The case of  $C_6$ -free graphs.* As it can be easily verified, the first inequality  $f(x, y) \geq 2f(x-2, y-7)$  can only occur when the algorithm branches according to Case 1(i) in the proof of Lemma 3, and in particular when  $P^+ \cup P^-$  is a cycle with *exactly* 6 edges. Now assume that the input Smith graph  $G$  does not contain any induced cycle of 6 vertices. Then the first of the above inequalities never occurs, and thus, in order to upper-bound the running time of the algorithm on  $C_6$ -free graphs, it suffices to solve the above recurrence system only for the inequalities  $2, 3, \dots, 9$ . To this end, we set again  $f(x, y) = 2^{\alpha x + \beta y}$ . As it can be easily verified, the values  $\{\alpha = 0.155615, \beta = 0.094385\}$  satisfy all these inequalities together. Therefore, as  $\alpha x + \beta y \leq (\alpha + \frac{3}{2}\beta)n = 0.2971925 \cdot n$ , and thus the running time of the algorithm on  $C_6$ -free graphs is upper-bounded by  $2^{0.2971925 \cdot n}$ .

*Part C: A more careful analysis of the worst case.* As it has been observed in Part A, by setting  $f(x, y) = 2^{\alpha x + \beta y}$ , the values  $\{\alpha = 0.15, \beta = 0.1\}$  satisfy the first two relations as equalities and all the others as strict inequalities. Thus the worst case in Part A can occur when only one of the first two recurrence relations is realized at every iteration of the algorithm. Assume that the first (resp. second) recurrence is applied in total in  $k$  (resp.  $\ell$ ) branching iterations of the algorithm. Then, at the end of the algorithm we have  $2^{k+\ell}$  produced instances. Recall by the analysis of Part B that the first inequality  $f(x, y) \geq 2f(x-2, y-7)$  can only occur when the algorithm branches according to Case 1(i) in the proof of Lemma 3, where  $P^+ \cup P^-$  is a cycle with *exactly* 6 edges. In this case, exactly four  $C_0$ -edges and three chords are being forced (i.e. colored). In addition, these four newly forced  $C_0$ -edges are either two pairs of consecutive  $C_0$ -edges, or three consecutive  $C_0$ -edges and one separate  $C_0$ -edge (i.e. not consecutive with the other three ones). Furthermore, as it can be easily verified, the second inequality  $f(x, y) \geq 2f(x-4, y-4)$  can only occur when the algorithm branches according to Case 1(iv) in the proof of Lemma 3. In this case, exactly three consecutive  $C_0$ -edges and one chord are being forced (i.e. colored). Then, after the  $k$  and  $\ell$  applications of the first and the second recurrence, respectively,  $4k + 3\ell$   $C_0$ -edges and  $3k + \ell$  chords have been forced. Since there are  $n$   $C_0$ -edges and  $\frac{n}{2}$  chords in  $G$ , it follows that  $4k + 3\ell \leq n$  and  $3k + \ell \leq \frac{n}{2}$ . From these inequalities it easily follows that the number  $2^{k+\ell}$  of produced instances is maximized when  $k = \frac{n}{10}$  and  $\ell = \frac{n}{5}$ , that is, when the algorithm branches  $\frac{n}{10}$  (resp.  $\frac{n}{5}$ ) times according to the first (resp. second) recursion rule. Only in this case the algorithm can produce in total  $\Theta(2^{\frac{n}{10} + \frac{n}{5}}) = \Theta(2^{0.3n})$  instances.

We will prove that this worst case cannot be realized. To this end, first recall by Case 1(iv) in the proof of Lemma 3 that the second inequality (i.e. when we have exactly 4 new biased vertices and exactly 4 new forced edges) is only possible to appear at a vertex  $v_y$  when, until the current iteration,  $v_y$  is in the center of a path of at least 6 consecutive  $C_0$ -edges that have not been colored yet. Then, in each of the two resulting new instances  $I_1, I_2$ , exactly three new consecutive  $C_0$ -edges are forced and  $v_y$  is an internal vertex of this path of the three  $C_0$ -edges. For simplicity of the presentation, let us call these three  $C_0$ -edges the *imperative path* of this application of the second recursion. Thus, in both  $I_1, I_2$ , between any two imperative paths there is at least one edge that does not belong to any imperative path. Furthermore, in order to achieve  $\frac{n}{10}$  applications of the first recursion and  $\frac{n}{5}$  applications of the second recursion, at the end of the algorithm every two consecutive imperative paths have to be separated by either one, two, or three consecutive  $C_0$ -edges which are forced by an application of the first reduction.

Now consider two consecutive imperative paths  $P_1, P_2$  in one of the instances at the end of the algorithm,

and assume that  $P_1, P_2$  are separated by the two consecutive  $C_0$ -edges  $v_{i-1}v_i, v_iv_{i+1}$ ; that is,  $P_1$  ends at vertex  $v_{i-1}$  and  $P_2$  ends at vertex  $v_{i+1}$ . Furthermore, assume without loss of generality that  $P_1$  has been forced during the algorithm *before*  $P_2$  has been forced. Then, since the edges  $v_{i-1}v_i, v_iv_{i+1}$  are forced later (i.e. after the edges of  $P_1, P_2$  have been forced) by an operation of the first recursion, these two edges have to belong to a cycle  $C_6$  with 6 edges. In particular, this  $C_6$  must contain the two chords  $v_{i-1}v_{i+1}^*$  and  $v_{i+1}v_{i-1}^*$ , while  $v_{i-1}^*$  and  $v_{i+1}^*$  must have distance exactly two along  $C_0$ . Let without loss of generality  $v_{i-1}^* = v_{k-1}$  and  $v_{i+1}^* = v_{k+1}$ . Furthermore, due to our assumption above that the first (resp. second) recursion is applied in total  $\frac{n}{10}$  (resp.  $\frac{n}{5}$ ) times during the algorithm, it follows that the edges  $v_{k-2}v_{k-1}$  and  $v_{k+1}v_{k+2}$  belong to two other imperative paths  $P_3$  and  $P_4$  of the second recursion, respectively. Note here that, according to our assumptions above,  $P_2 = (v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4})$  and  $P_4 = (v_{k+1}, v_{k+2}, v_{k+3}, v_{k+4})$ , where  $v_{i+1}^* = v_{k+1}$ . Furthermore note that some of the paths  $P_1, P_2$  may coincide with some of the paths  $P_3, P_4$ .

Assume that the imperative path  $P_4$  is forced *before* the imperative path  $P_2$  is forced. Then, at the time when the algorithm branches to the second instance in the application of the second reduction at  $P_2$ , the imperative path that replaces  $P_2$  is either the path  $(v_i, v_{i+1}, v_{i+2}, v_{i+3})$  or the path  $(v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5})$ . If  $P_2$  is replaced by the path  $(v_i, v_{i+1}, v_{i+2}, v_{i+3})$ , then the  $C_6$  that corresponds to the application of the first reduction which forces the edge  $v_{i-1}v_i$  must also contain the three consecutive  $C_0$ -edges  $v_{k-1}v_k, v_kv_{k+1}, v_{k+1}v_{k+2}$ . This is a contradiction, as the edge  $v_{k+1}v_{k+2}$  belongs to the imperative path  $P_4$  which has been forced earlier during the algorithm. On the other hand, if  $P_2$  is replaced by the path  $(v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5})$ , then the  $C_6$  that corresponds to the application of the first reduction which forces the edge  $v_{i-1}v_i$  must contain the three consecutive  $C_0$ -edges  $v_{i-1}v_i, v_iv_{i+1}, v_{i+1}v_{i+2}$ , as well as the  $C_0$ -edge  $v_{k-1}v_k$ , but *not* the  $C_0$ -edge  $v_kv_{k+1}$ . That is, in this case the edge  $v_kv_{k+1}$  belongs neither to an application of the first nor to an application of the second recurrence.

In a similar way one can also analyze the case where the imperative path  $P_4$  is forced *after* the imperative path  $P_2$  is forced, as well as the cases where between the imperative paths  $P_1, P_2$  there are three  $C_0$ -edges or just one  $C_0$ -edge. In each of these cases, it follows that in some of the produced instances, the algorithm either forces a specific  $C_0$ -edge in both an application of the first and the second recursion (which is a contradiction), or misses one  $C_0$ -edge (i.e. it does not force it with an application of either the first or the second recursion). That is, when the algorithm only applies the first two reductions (which is the worst case as discussed above), after every at most four applications of the second reduction (i.e. the ones that create the imperative paths  $P_1, P_2, P_3, P_4$ ), one  $C_0$ -edge is not being forced in some of the produced instances, and thus the algorithm can never produce in total  $\Theta(2^{0.3n})$  instances. That is, there exists a strictly positive  $\varepsilon > 0$  such that the algorithm produces at most  $O(2^{(0.3-\varepsilon)n})$  instances. This completes the analysis of Part C of the proof.

Finally, assume that we have already computed all the desired alternating red-blue *cycles*  $D = \text{Red} \cup \text{Blue}$  of  $I$ , each of which, using the ambivalent quadruples, can potentially encode many other alternating red-blue cycles. It remains to compute the running time of each execution of Procedure 2. The symmetric difference  $C_0 \Delta D$  can be computed in linear time. Assume that  $C_0 \Delta D$  has  $k \geq 2$  connected components. Then we create in linear  $O(n)$  time a new auxiliary graph  $H = (V_H, E_H)$ , as follows. The vertex set  $V_H$  has one vertex for every connected component (i.e. cycle) of  $C_0 \Delta D$ , and thus  $|V_H| \leq n$ . Let  $u_1, u_2 \in V_H$  be two different vertices of  $H$ , i.e. corresponding to two different connected components of  $C_0 \Delta D$ . Then,  $u_1$  is adjacent to  $u_2$  in  $E_H$  if and only if there exists an ambivalent quadruple which, if flipped, will connect the two corresponding connected components of  $C_0 \Delta D$ . Lines 1-3 of Procedure 2 can be implemented as follows. We run any linear-time (i.e.  $O(n)$ -time) connectivity algorithm on  $H$  such as Breadth-First-Search. If  $H$  is not connected then no sequence of flips of the ambivalent quadruples can connect the components of  $C_0 \Delta D$  into one Hamiltonian graph of  $G$ . Otherwise we obtain a spanning tree of  $H$ , and in this case the edges of the spanning tree indicate those ambivalent quadruples that need to be flipped in order to make  $C_0 \Delta D$  a Hamiltonian graph of  $G$ . ■



## 5 Efficiently computing another long cycle in a Hamiltonian graph

In this section we prove our results on approximating the length of a second cycle on graphs with minimum degree  $\delta \geq 3$  and maximum degree  $\Delta$ . In [1], Bazgan, Santha, and Tuza considered the optimization problem of efficiently (i.e. in polynomial time) constructing a large second cycle different than the given Hamiltonian cycle  $C_0$  in a given Hamiltonian graph  $G$ . In particular they proved the following results.

**Theorem 4 ([1])** *Let  $G$  be an  $n$ -vertex cubic Hamiltonian graph and let  $C_0$  be a Hamiltonian cycle of  $G$ . Given  $G$  and  $C_0$ , for every  $\varepsilon > 0$ , a cycle  $C' \neq C_0$  of length at least  $(1-\varepsilon)n$  can be found in time  $2^{O(1/\varepsilon^2)} \times n$ .*

**Theorem 5 ([1])** *Let  $G$  be an  $n$ -vertex cubic Hamiltonian graph and let  $C_0$  be a Hamiltonian cycle of  $G$ . There is an algorithm which, given  $G$  and  $C_0$ , computes a cycle  $C' \neq C_0$  of length at least  $n - 4\sqrt{n}$  in time  $O(n^{3/2} \log n)$ .*

### 5.1 Notation and preliminary results

Before we proceed to the main result of the section, we introduce some necessary notation and state preliminary results. Let  $G = (V, E)$  be a graph with a designated Hamiltonian cycle  $C_0 = (v_1, v_2, \dots, v_n, v_1)$ . Two chords of  $C_0$  are *independent* if they do not share an endpoint. The *length* of a chord  $v_i v_j$ , with  $i < j$ , is defined as  $\min\{j - i, n + i - j\}$ . We say that two vertices  $u, v \in V$  are *chord-adjacent* if they are connected by a chord of  $G$ . Two independent chords  $e_1$  and  $e_2$  are called *crossing* if their endpoints appear in an alternating order around  $C_0$ ; otherwise  $e_1$  and  $e_2$  are called *parallel*.

For  $x, y \in V$ , we denote by  $d(x, y)$  the length of the path from  $x$  to  $y$  around  $C_0$ . Note that, in general,  $d(x, y) \neq d(y, x)$ . We define the *distance* between two independent chords  $xy$  and  $ab$  as follows:

1. if  $xy$  and  $ab$  are crossing, such that  $a$  lies on the path from  $x$  to  $y$  around  $C_0$ , then  $\text{dist}(xy, ab) = \min\{d(x, a) + d(y, b), d(b, x) + d(a, y)\}$ ;
2. if  $xy$  and  $ab$  are parallel such that neither  $y$  nor  $b$  lie on the path from  $x$  to  $a$  around  $C_0$ , then  $\text{dist}(xy, ab) = d(x, a) + d(b, y)$ .

In the proof of our main result of this section (see Theorem 6) we use the following two lemmas. The first one is a basic fact from graph theory and the second one is straightforward to check (see Fig. 2 for an illustration).

**Lemma 4** [[18], Exercise 3.1.29] *Let  $G = (V, E)$  be a bipartite graph of maximum degree  $\Delta$ . Then  $G$  has a matching of size at least  $\frac{|E|}{\Delta}$ .*

**Lemma 5** *Let  $G = (V, E)$  be an  $n$ -vertex graph with a Hamiltonian cycle  $C_0$ .*

- (1) *If  $G$  has a chord of length  $\ell$ , then  $G$  contains a cycle  $C' \neq C_0$  of length at least  $n - \ell + 1$ .*
- (2) *If  $G$  has two crossing chords  $e_1, e_2$  and  $\text{dist}(e_1, e_2) = d$ , then  $G$  contains a cycle  $C' \neq C_0$  of length at least  $n - d + 2$ .*
- (3) *If  $G$  has four pairwise independent chords  $e_1, e_2, f_1$ , and  $f_2$  such that*
  - (a)  *$e_1, e_2$  are parallel and  $f_1, f_2$  are parallel,*
  - (b)  *$e_i$  and  $f_j$  are crossing for every  $i, j \in \{1, 2\}$ ,*
  - (c)  *$\text{dist}(e_1, e_2) = d_1$  and  $\text{dist}(f_1, f_2) = d_2$ ,*

*then  $G$  contains a cycle  $C' \neq C_0$  of length at least  $n - d_1 - d_2 + 4$ .*

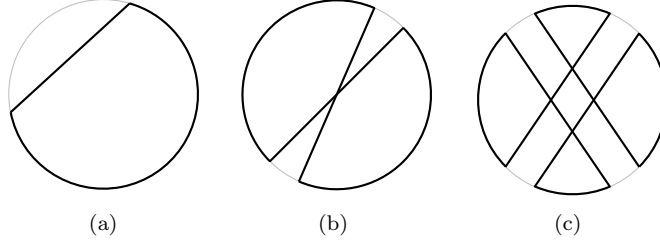


Figure 2: An illustration of Lemma 5. (a) A short chord. (b) A pair of crossing chords. (c) Crossing pairs of parallel chords.

## 5.2 Long cycles in Hamiltonian graphs

**Theorem 6** *Let  $G = (V, E)$  be an  $n$ -vertex Hamiltonian graph of minimum degree  $\delta \geq 3$ . Let  $C_0 = (v_1, v_2, \dots, v_n, v_1)$  be a Hamiltonian cycle in  $G$  and let  $\Delta$  denote the maximum degree of  $G$ . Then  $G$  has a cycle  $C' \neq C_0$  of length at least  $n - 4\alpha(\sqrt{n} + 2\alpha) + 8$ , where  $\alpha = \frac{\Delta-2}{\delta-2}$ . Moreover, given  $C_0$ , such a cycle  $C'$  can be computed in  $O(m)$  time, where  $m = |E|$ .*

**Proof.** We start by showing the existence of the desired cycle  $C'$ . Without loss of generality we assume that  $\alpha < \frac{\sqrt{n}}{2}$ , as otherwise any cycle  $C' \neq C_0$  in  $G$  satisfies the theorem. Furthermore, we assume that the length of every chord in  $G$  is at least  $4\alpha(\sqrt{n} + 2\alpha) - 6$ , as otherwise the existence of  $C'$  follows from Lemma 5 (1).

Let  $q = \alpha\sqrt{n}$ . We partition arbitrarily the vertices of the Hamiltonian cycle  $C_0$  into  $r$  consecutive intervals  $B_0, B_1, \dots, B_{r-1}$ , such that  $r \in \left\{ \left\lfloor \frac{\sqrt{n}}{\alpha} \right\rfloor, \left\lfloor \frac{\sqrt{n}}{\alpha} \right\rfloor + 1 \right\}$  and  $|q| \leq |B_i| \leq |q| + 2\alpha^2$  for every  $i \in \{0, 1, \dots, r-1\}$ . It is a routine task to check that such a partition exists.

For every  $i \in \{0, 1, \dots, r-1\}$  we denote by  $W_i$  the set of vertices that are chord-adjacent to a vertex in  $B_i$ , and by  $E_i$  we denote the set of chords that are incident to a vertex in  $B_i$ . Furthermore, we denote by  $H_i$  the graph with vertex set  $B_i \cup W_i$  and edge set  $E_i$ . Since the length of every chord in  $G$  is at least  $4\alpha(\sqrt{n} + 2\alpha) - 6$ , observe that for every  $i \in \{0, 1, \dots, r-1\}$ , the set  $W_i$  is disjoint from  $B_{i-1} \cup B_i \cup B_{i+1}$  (where the arithmetic operations with indices are modulo  $r$ ). The latter, in particular, implies that  $H_i$  is a bipartite graph with color classes  $B_i$  and  $W_i$ .

Let  $i, j \in \{0, 1, \dots, r-1\}$  be two distinct indices, we say that the intervals  $B_i$  and  $B_j$  are *matched* if there exist two independent chords such that each of them has one endpoint in  $B_i$  and the other endpoint in  $B_j$ . We claim that every interval  $B_i$  is matched to another interval  $B_j$  for some  $j \in \{0, 1, \dots, r-1\} \setminus \{i-1, i, i+1\}$ . Indeed, by Lemma 4, graph  $H_i$  has a matching  $M_i$  of size at least

$$\frac{|q|(\delta-2)}{\Delta-2} = \frac{\lfloor \alpha\sqrt{n} \rfloor}{\alpha} > \frac{\alpha\sqrt{n}-1}{\alpha} \geq \sqrt{n}-1 > \left\lfloor \frac{\sqrt{n}}{\alpha} \right\rfloor - 2 \geq r-3,$$

and therefore, by the pigeonhole principle, there exists  $j \in \{0, 1, \dots, r-1\} \setminus \{i-1, i, i+1\}$  such that at least two edges in  $M_i$  have their endpoints in  $B_j$ , meaning that  $B_i$  is matched to  $B_j$ .

Let  $\sigma : \{0, 1, \dots, r-1\} \rightarrow \{0, 1, \dots, r-1\}$  be a function such that  $B_i$  is matched to  $B_{\sigma(i)}$ , and denote by  $f_{i,1}$  and  $f_{i,2}$  some fixed pair of independent chords between  $B_i$  and  $B_{\sigma(i)}$ . We observe that  $\text{dist}(f_{i,1}, f_{i,2}) \leq 2(|q| + 2\alpha^2 - 1) \leq 2\alpha(\sqrt{n} + 2\alpha) - 2$ , as the endpoints of  $f_{i,1}$  and  $f_{i,2}$  lie in the intervals  $B_i$  and  $B_{\sigma(i)}$  each of length at most  $|q| + 2\alpha^2$ .

Let now  $R$  be an auxiliary graph with a Hamiltonian cycle  $(x_0, x_1, \dots, x_{r-1})$  and the chord set being  $\{x_i x_{\sigma(i)} : i = 0, 1, \dots, r-1\}$ . Let  $x_i x_j$  be a chord in  $R$  of the minimum length, where  $j = \sigma(i)$ . Without loss of generality, we assume that  $i < j$  and  $j - i \leq r + i - j$ . Let  $x_k$  be a vertex of  $R$  such that  $i < k < j$  and let  $s = \sigma(k)$ . Since  $x_i x_j$  is of minimum length, the chords  $x_i x_j$  and  $x_k x_s$  are crossing, and hence each of  $f_{i,1}$  and  $f_{i,2}$  crosses both  $f_{k,1}$  and  $f_{k,2}$ .

Finally, if  $f_{i,1}, f_{i,2}$  or  $f_{k,1}, f_{k,2}$  are crossing, then by Lemma 5 (2) there exists a cycle  $C' \neq C_0$  of length at least  $n - 2\alpha(\sqrt{n} + 2\alpha) + 4$ . Otherwise,  $f_{i,1}, f_{i,2}$  are parallel and  $f_{k,1}, f_{k,2}$  are parallel, and hence by Lemma 5

(3) there exists a cycle  $C' \neq C_0$  of length at least  $n - 4\alpha(\sqrt{n} + 2\alpha) + 8$ , which proves the first part of the theorem.

The above proof is constructive. We now explain at a high level how the proof can be turned into the desired algorithm. First, if  $\alpha \geq \frac{\sqrt{n}}{2}$ , then we output any cycle formed by a chord and the longer path of  $C_0$  connecting the endpoints of the chord. Otherwise, we partition the vertices of  $C_0$  into the intervals  $B_1, \dots, B_{r-1}$  and we assign to each vertex the index of its interval. Clearly, this can be done in  $O(n)$  time. Next, we traverse the vertices of  $G$  along the cycle  $C_0$  and for every vertex  $v$  of an interval  $B_i$  we check the chords incident to  $v$ . If we encounter a chord  $f$  of length less than  $4\alpha(\sqrt{n} + 2\alpha) - 6$ , then we output the cycle formed by  $f$  and the longer path of  $C_0$  connecting the endpoints of  $f$ . Otherwise, for the interval  $B_i$  we keep the information of how many and which vertices of  $W_i$  belong to other intervals  $B_j$  for  $j \in \{0, 1, \dots, r-1\} \setminus \{i-1, i, i+1\}$ . When we find an interval  $B_j$  that has at least two elements from  $W_i$ , we set  $\sigma(i)$  to  $j$  and proceed to the first vertex of the next interval  $B_{i+1}$ . By doing this, we also keep the information of the current shortest chord in the graph  $R$  (defined in the proof above). After finishing this procedure: (1) we have a function  $\sigma(\cdot)$ ; (2) for every  $i \in \{0, 1, \dots, r-1\}$  we know a pair  $f_{i,1}, f_{i,2}$  of independent edges between  $B_i$  and  $B_{\sigma(i)}$ ; and (3) we know  $k$  such that  $x_k x_{\sigma(k)}$  is a minimum length chord in  $R$ . Clearly, this information is enough to identify the desired cycle in constant time. In total, we spent  $O(n)$  time to compute the partition of the vertices into the intervals and we visited every chord at most twice, which implies the claimed  $O(m)$  running time. ■

The next two corollaries are implied as immediate consequences of Theorem 6, and they provide immediate extensions of the results of [1] and [9], respectively.

**Corollary 2** *Let  $G = (V, E)$  be an  $n$ -vertex Hamiltonian  $\delta$ -regular graph with  $\delta \geq 3$ , and let  $C_0$  be a Hamiltonian cycle of  $G$ . Then  $G$  has a cycle  $C' \neq C_0$  of length at least  $n - 4\sqrt{n}$ , which can be computed in  $O(\delta n)$  time.*

**Corollary 3** *Let  $G = (V, E)$  be an  $n$ -vertex Hamiltonian graph of minimum degree  $\delta \geq 3$ . Let  $C_0$  be a Hamiltonian cycle of  $G$  and let  $\Delta$  denote the maximum degree of  $G$ . If  $\frac{\Delta}{\delta} = o(\sqrt{n})$ , then  $G$  has a cycle  $C' \neq C_0$  of length at least  $n - o(n)$ , which can be computed in  $O(m)$  time.*

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